CAYLEY OCTONIONS AND STRONG INTERACTIONS

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(presented by D. KASTLER)

Let $\underline{Q}$ be the 4-dimensional real space (and field) of quaternions. j_1, j_2, j_3 being the quaternionic units, a —— operation is defined on $\underline{Q}$ by putting

$$\overline{x^0 + x^1 j_1 + x^2 j_2 + x^3 j_3} = x^0 - x^1 j_1 - x^2 j_2 - x^3 j_3 .$$

$\underline{Q}$ can then be written as the direct sum $\underline{Q} = \underline{Q}_s \oplus \underline{Q}_v$ of the one-dimensional real space $\underline{Q}_s$ of self-adjoint (scalar) quaternions and the three-dimensional real space $\underline{Q}_v$ of antiself-adjoint (vectorial) quaternions. Putting $j_k = i\,\sigma_k$ where the σ_k are Pauli's matrices, quaternions can be considered as a set of 2 × 2 complex matrices, —— being then the usual operation of taking the adjoint matrix. $\underset{\approx}{Q}$ is an euclidian space under the norm $\|q\|$ given by $\|q\|^2 = \bar{q}\, q = x^{0\,2} + x^{1\,2} + x^{2\,2} + x^{3\,2}$, the corresponding real scalar product being $(q,\, q') = \frac{1}{2}\,\mathrm{Tr}(\bar{q}\, q') = \frac{1}{2}\,(\bar{q}\, q' + \bar{q}'\, q)$. Each complex 2 × 2 matrix can be uniquely written as $q + iq'$, q and q' being quaternions.

$\underline{M}$ shall be a 7-dimensional real euclidian charge-space for mesons, taken as the direct sum $\underset{\approx}{M} = \underline{Q}_v \oplus \underline{Q}$ of a 3-dimensional pionic subspace isomorphic to $\underline{Q}_v$ and a 4-dimensional kaonic subspace isomorphic to $\underline{Q}$. To assign particles on a precise way, the general element M of $\underline{M}$ shall be taken as $M = \pi \oplus K$, π and K being the quaternions (written as 2 × 2 matrices) :

$$\pi = \begin{pmatrix} \pi^0 & -\sqrt{2}\,\pi^+ \\ \sqrt{2}\,\pi^- & -\pi^0 \end{pmatrix} \qquad K = \sqrt{2}\begin{pmatrix} \bar{K}^0 & -K^+ \\ K^- & K^0 \end{pmatrix}$$

(we have $\pi^0 = i\pi^3$).

The baryon charge space shall be the (8 dimensional complex) spin space of $\underline{M}$, taken as the complex extension $\underline{B} + i\underline{B}$ of an 8-dimensional euclidian real space $\underline{B}$ which we shall describe as the direct sum $\underline{B} = \underline{Q} \oplus \underline{Q}$ of two 4-dimensional real spaces both isomorphic to $\underline{Q}$ and corresponding respectively to nucleon - Ξ and Σ - Λ : precisely, the most general baryon charge state B shall be written as $B = N \oplus \Sigma$ where N and Σ are the 2 × 2 matrices (not quaternions)

$$N = \sqrt{2}\begin{pmatrix} \Xi^0 & p \\ \Xi^- & n \end{pmatrix} \qquad = \begin{pmatrix} \Lambda^0 + \Sigma^0 & -\sqrt{2}\,\Sigma^+ \\ \sqrt{2}\,\Sigma^- & \Lambda^0 - \Sigma^0 \end{pmatrix}$$

(we have $\Sigma^0 = i\,\Sigma^3$)

The different kinds of invariance for strong interactions considered in the literature can then be described on the following simple fashion : the basic Clifford matrix ρ for $\underline{M}$, related to the Cayley octonians, is a linear operator on $\underline{B}$ depending linearly on the elements M of $\underline{M}$ (the ρ's are the equivalents for $\underset{\approx}{M}$ of Dirac's γ's for Minkowski space). As $\underline{B} = \underline{Q} \oplus \underline{Q}$, ρ can be written as a 2 × 2 matrix of linear operators on $\underline{Q}$, namely

$$\rho\,(M) = \rho\,(\pi \oplus K) = \begin{pmatrix} L(\pi) & R(K) \\ -R(\bar{K}) & L(\bar{\pi}) \end{pmatrix}$$

where $L(q)$ and $R(q)$ denote the linear operators on $\underline{Q}$ obtained respectively by multiplying left or right by the quaternion q. (note that $\bar{\pi} = -\pi$). It is easily verified that these ρ - matrices satisfy the anticommutation rules

$$\{\rho(M),\ \rho(M')\}_+ = -2(M,\ M') = -\ 2(\pi,\pi') - 2(K,\ K').$$

Λ being now an orthogonal transformation of $\underline{M}$ of determinant + 1 the corresponding spin transformation $\mathcal{S}[\Lambda]$ of B is defined by the covariance rule

$$\rho(\Lambda\ M) = \mathcal{S}[\Lambda]\ \rho\ (M)\ \mathcal{S}[\Lambda]^{-1}$$

All candidates for the invariance group of strong interactions are some subgroups of the special orthogonal group on $\underline{M}$ accompanied by the corresponding spin subgroups on $\underline{B}$. Usually one considers only subgroups already contained in the subgroup $\mathcal{G}$ which let invariant the two subspaces $\underline{Q}_v$ and $\underline{Q}$ of $\underline{M}$ corresponding to π and K mesons. The corresponding Λ's and their induced $\mathcal{S}[\Lambda]$'s on $\underline{B}$ can explicitely be described by means of multiplications by quaternions : they are easily seen to be of the form

$$\Lambda : \begin{cases} \pi \longrightarrow R\pi R^{-1} \\ K \longrightarrow QK\ P^{-1} \end{cases} \qquad \mathcal{S}[\Lambda] : \begin{cases} N \longrightarrow RNP^{-1} \\ \Sigma \longrightarrow R\Sigma Q^{-1} \end{cases}$$

where P, Q and R are arbitrary quaternions of norm 1.

$\mathcal{G}$ is thus seen to be the product of 3 commuting subgroups each of them isomorphic to the group of unitary quaternions (i.e. the covering group of the group of rotations in 3 dimensions).

a) The group of isotopic spin transformation is obtained by taking Q = R, P = I.

b) The hypercharge Y is defined by taking the infinitesimal transformation Q = R = I $P = I + \varepsilon j_3$.

c) Electromagnetic gauge transformation is obtained by putting $P = Q = R = e^{\frac{1}{2}\alpha\, j_3}$.

Considering only π-interactions we have :

d) global invariance P, Q, R arbitrary ;

e) restricted invariance Q, R arbitrary, $P = e^{\varphi j_3}$.

f) Taking P = R = I, Q arbitrary, one obtains Tiomno's doublet approximation.

g) As a possible invariance group not contained in $\mathcal{G}$ we refer to J.M. Souriau [2].

The most symmetric lagrangian for strong interactions is

$$\mathcal{L} = i(B,\ \rho\ (M)\ B) = \frac{i}{2}\ \mathrm{Tr}(\bar{N}\ \pi\ N - \bar{\Sigma}\ \pi\ \Sigma + \Sigma\ K\ \bar{N} - N\ \bar{K}\ \bar{\Sigma}).$$

It is evident in this formalism that $\mathcal{L}$ is invariant by the full special orthogonal group of the seven dimensional $\underline{M}$. $\mathcal{L}$ is a possible candidate for the leading part of strong interactions at ultra high energies.

REFERENCES

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