IL NUOVO CIMENTO VOL. XXX, N. 2 16 Ottobre 1963

Five-Dimensional Relativity.

J.-M. SOURIAU

Faculté des Sciences, Université d'Aix - Marseille

(ricevuto il 13 Maggio 1963)

Summary. — We study the hypothesis where the universe U is a five-dimensional Riemannian manifold, wich satisfies certain global topological conditions. We postulate the existence of a principle of relativity wich treats on equal basis the five dimensions of U; the laws wich satisfy this principle have an approximate description in a 4-dimensional space-time manifold $\hat{U}$; this gives the possibility of comparing them with the usual description of experimental laws. Thus, if we extend to the fifth dimension the invariance of general relativity, we obtain *classical electrodynamics*: the equations of Maxwell, conservation of electricity, electromagnetic forces, etc. Likewise, the five-dimensional extension of the invariance of the wave equations leads one automatically to electromagnetic terms, such as they are actually observed; the electric charge, for instance, is found to be an integral multiple of an *elementary charge* which depends neither on the mass, nor on the spin. Among the other consequences of the theory, we find *gauge invariance*, and *charge conjugation*; the *maximum violation of parity* in β-decays; the existence of *two neutrinos* of opposite chirality.

1. – Introduction.

One usually classifies the forces (or «interactions») which are found in nature in four types:

I) Gravitational interactions II) Electromagnetic interactions	long range ($> 10^{19}$ cm)
III) Weak interactions IV) Strong interactions	short range ($< 10^{-11}$ cm).

5123

The purpose of a unitary theory is to give a common geometrical descriptionn to all the forces of nature, in the same way as general relativity « explains » the forces of gravitation through the curvature of space-time.

In the 1920's, since the short-range forces were still unknown, the unitary theories aimed at unifying the electromagnetic and the gravitational field; already in 1921, Kaluza's theory (1) gave results of this type.

Today, two directions of research seem to be offered:

1) to construct a strictly unitary theory which can explain both long- and short-range forces;

2) to construct a theory which explains long-range forces, and gives a geometrical frame for the description of elementary particles and their short-range interactions.

Now, it is not impossible that five dimensional relativity should allow one to achieve this second program.

The idea of adding a fifth dimension to space-time appears in many authors; either to simplify the study of spinors (2), or to give an interpretation to the Hamiltonian action (3), or else in Kaluza's theory (1).

If such a method is to be more than a simple mathematical trick, it is necessary to put forward a symmetry, as large as possible, between the five dimensions.

In Kaluza's theory, one considers a five-dimensional riemannian manifold, where equations similar to Einstein's are satisfied; the fifteen equations thus obtained satisfy indeed this symmetry; but the symmetry is broken by two supplementary postulates of the theory: a principle of « stationarity » is added in which the fifth dimension plays a particular role; one of the fifteen equations is also modified, in a nonsymmetrical way.

In spite of these epistemological shortcomings, Kaluza's theory contains some remarquable results: the fourteen equations which are left turn out to be the usual equations of Einstein and Maxwell; the theory gives a geometrical origin to the principle of electromagnetic gauge invariance—which principle is of purely phenomenological origin in ordinary relativity, but is satisfied in all its physical consequences, including those which pertain to the domain of elementary particles physics.

It would thus be tempting to try to keep those results while discarding the hypotheses which break the symmetry between the five dimensions; the theory of Jordan and Thiry (4) for instance, uses in a symmetrical way the fifteen

---

(1) Th. Kaluza: *Sitz. Preus. Akad.*, 966 (1921).

(2) G. Petiau: *Journ. Math. Pures et Appl.*, **9**, 1 (1947).

(3) J.-M. Souriau: *Coll. Blaise Pascal* (Clermont-Ferrand, 1962).

(4) Y. Thiry: *Journ. Math. Pures et Appl.*, **9**, 275 (1951).

field equations; unfortunately, this theory keeps the principle of stationarity where the components g_{jk} of the fundamental tensor (*) are independent of the fifth co-ordinate x^5. As a consequence of this principle, the five-dimensional universe U acquires a structure of *bundle space*; its *basis* $\hat{U}$ is a four-dimensional Riemannian manifold, which is naturally identified with space-time; one can give an algorithm wich describes each field of U by means of fields of $\widehat{U}$ (see the Appendix): thus, the theory has a quadridimensional formulation, which allows one to recognize, among others, the gravitational and the electromagnetic fields.

A priori, it would seem that this principle of stationarity is necessary to match the theory with experiment. But it breaks the symmetry between the five co-ordinates, since in the long run, this theory only possesses the invariance of general relativity and the invariance in the gauge transformations

(1) $$x^5 \to x^5 - f(x^\mu)\,. \qquad \text{(see note (*))}\,,$$

Now, in 1926, KLEIN [6,7] suggested among other hypotheses, to replace the condition of stationarity by the following one:

(2) — the g_{jk} are to be *periodical* functions of x^5 —

This condition, much weaker than the preceding one, does not seem to be sufficient to explain why the universe appears to have only four dimensions; this is perhaps the reason why, in an article published in 1938 [5], EINSTEIN and BERGMANN suggested to add other conditions (which would rather tend toward the theory of Kaluza).

However, in 1958, PAULI ([8], note 23) suggested to go back to Klein's original idea and to study, on the five-dimensional manifold, other fields than the tensor field g_{jk} (in particular a spinor field). Now the theory which we suggested independently in 1958 realizes this program of PAULI, and gives it a more precise meaning (**).

Indeed, it is not difficult to recognize, in Klein's condition (2), an hypothesis on the topological nature of the universe U; this condition, indeed, suggests that the fifth dimension is closed upon itself, that is to say, that U

(*) We will denote by latin letters ($j, k, \dots$) the indices which take the values 1, 2, 3, 4, 5; by greek letters ($\mu, \nu, \dots$) those which take the values 1, 2, 3, 4.

(**) Ref. [9]. For a more detailed account of the theory, see [10], Chapter VII.

(5) A. EINSTEIN and P.-G. BERGMANN: *Ann. Math.*, **2**, 39, 683 (1938).

(6) O. KLEIN: *Zeits. fur Phys.*, **37**. 895 (1926).

(7) O. KLEIN: *Nature*, **118**, 516 (1927).

(8) W. PAULI: *Theory of Relativity* (London, 1958).

(9) J.-M. SOURIAU: *Compt. Rend.*, **247**, 1559 (1958).

(10) J.-M. SOURIAU: *Géométrie et relativité* (Paris, 1963).

is homeomorphic to a « tube », direct product of space-time (which is supposed to be simply connected) by a circle, and that the x^j are the co-ordinates of the universal covering of U.

When x^5 increases by one period, one arrives at the *same point* of U; all the fields, therefore, take on the same value (and not only the field g_{jk}).

We should note that, in his account of the theory of Jordan-Thiry ([11]), LICHNEROWICZ gave this circular structure to the fifth dimension; it can be shown that this circle is spacelike (*); we will note its length $2\pi\xi$.

Jordan and Thiry's theory does not give a way to calculate the value of ξ (**), but nothing prevents one to suppose that ξ is *very small*, and it is clear that this condition by itself is sufficient to explain the quadri-dimensional appearence of the universe (***); then the hypothesis of stationarity is no longer necessary, and it is natural to try to do without it, as we have done. One thus finds a theory which is invariant in all the *local* transformations of the form

$$x^j \to f^j(x^k) , \tag{3}$$

where the f^j are arbitrary functions (***); the topological structure attributed to U plays no role in the field equations, but only in the *global* transformations of U. The theory of covering space (see ([10]), propositions (10.29) and (14.42)) shows that these transformations can be divided in *two classes*; one, which is a subgroup, contains the transformations of the form

$$x^\mu \to f^\mu(x^j) , \quad x^5 \to x^5 + f(x^j) \quad (f^\mu \text{ and } f \text{ have the same period in } x^5), \tag{4}$$

which have been considered by KLEIN and PAULI (([8]), note 23, formula (29)); we suggested to interpret the transformations of the second class (for instance the substitution $x^5 \to -x^5$) as *charge conjugations* ([12]); in the case of special relativity (zero curvature), the group of global isomorphisms of the riemannian structure of U is the direct product of the usual nonhomogeneous Lorentz group L by the complete orthogonal group in two dimensions O^2. Now MICHEL

(*) The inverse hypothesis, which has been put forward by some authors, leads to electrostatic forces which are attractive between charges of the same sign.

(**) In what follows we will give its value.

(***) One can imagine, if one wishes, that each point of space-time is in fact a submicroscopic circle of radius ξ; but this picture, like that of the tube, rather hampers the mind; and it is safer to rely only on differential geometry and on topology.

(***) Under the same restrictions as in general relativity: the transformation (3) must be differentiable a sufficient number of times, as also its inverse transformation.

([11]) A. LICHNEROWCIZ: *Théories relativistes de la gravitation et de l'électromagnétisme* (Paris, 1955).

([12]) J.-M. SOURIAU: *Compt. Rend.*, **248**, 1478 (1959).

5126

noticed ([13]) that elementary particles can be characterized by an irreducible representation of this group $L \times O^2$, which confirms this interpretation.

Our theory is fully pentadimensional (see formula (3)); it possesses however those of Jordan-Thiry and of Kaluza as approximations—in the same way as a tube can be regarded as a line and described in one dimension providing it is fine enough; or as a periodical medium (crystal) can be considered as homogeneous when its period is small enough.

This approximations are useful for the physical interpretation of the theory, as they allow one to give an approximate quadrimensional picture of it; but the true program of the theory consists in giving a penta-dimensional description of physics; the *principle of relativity in five dimensions*—according to which the equations of physics must be invariant in the transformations (3) imposes very strict conditions which it is possible to compare with experiment.

2. – The case of nonquantum relativity.

We will derive the field equations on the manifold U from a variational principle; the principle of 5-dimensional relativity is expressed by writing that the lagrangian density is *invariant* in the transformations (3).

The methods of differential geometry allow one to deduce equations analogous to those of general relativity:

1) To each physical phenomenon is associated a symmetrical tensor T_{jk} whose divergence (in the riemannian sense) is automatically zero.

2) The variation equations of the fundamental tensor g_{jk} give the pentadimensional Einstein's equation

$$R_{jk} - \tfrac{1}{2} R g_{jk} + \Lambda g_{jk} = \chi \sum T_{jk} , \tag{5}$$

where Λ and χ denote universal constants and where the sum in the right-hand side of the equation is extended to all phenomena which are present.

The *interpretation* of this results can be given in the approximation of Kaluza: we first give the approximation corresponding to Jordan-Thiry's theory and we then suppose the « radius of the tube » ξ to be constant.

We then describe the field equations by means of transverse variables

---

([13]) L. Michel: *Nuovo Cimento*, 9, 319 (1953).

(see the Appendix) $\tilde{g}_{\mu\nu}$, $\tilde{R}_{\mu\nu}$, $\tilde{T}_{\mu\nu}$, etc.; we also introduce the quantities

$$A_\mu = \xi \sqrt{\frac{2\pi}{\chi}}\, \mathscr{A}_\mu\,, \qquad \left[\mathscr{A}_\mu = \frac{g_{\mu 5}}{g_{55}}\right], \tag{6}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,, \tag{7}$$

$$J_\mu = \frac{1}{\xi}\sqrt{\frac{\chi}{2\pi}}\, \tilde{T}_{\mu 5}\,. \tag{8}$$

The tensors A_μ, $F_{\mu\nu}$, J_μ will be interpreted respectively as electromagnetic potential, electromagnetic field, electric current; the five-dimensional equation $\operatorname{div} T = 0$ can be divided quadri-dimensionally into

$$[\operatorname{div} \tilde{T}]_\mu + F_{\mu\nu} J^\nu = 0\,, \tag{9}$$

$$\operatorname{div} J = 0\,; \tag{10}$$

(10) expresses the conservation of electricity; (9) is the expression of the conservation of energy and momentum, taking into account the electromagnetic forces $F_{\mu\nu} J^\nu$; (9) therefore expresses the principle according to which electric charges are *attached to matter.*

Likewise eq. (5) can be divided quadridimensionally; one first obtains the 10 equations

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{R}\tilde{g}_{\mu\nu} + \Lambda \tilde{g}_{\mu\nu} = \chi\left[\Theta_{\mu\nu} + \sum \tilde{T}_{\mu\nu}\right], \tag{11}$$

where $\Theta_{\mu\nu}$ denote Maxwell's momentum-energy tensor built up from the $F_{\mu\nu}$; we recognize Einstein's equation (with the cosmological constant Λ and Einstein's constant χ) which allows one to consider all the forms of energy as sources of the gravitational field. (5) also gives the 4 equations

$$[\operatorname{div} F]_\mu = 4\pi \sum J_\mu \tag{12}$$

which constitute the *second group of Maxwell's equations* (the first group being an immediate consequence of (7)); these equations express as before that the sources of electromagnetic fields are all the forms of electricity.

Finally, in the approximation of Jordan-Thiry, we have a fifteenth equation (which is replaced in Kaluza's approximation by the equation $\xi = \text{const}$), which involves a new quantity $r = \tilde{T}_{55}/\xi^2$, of which we now give an interpretation.

To this effect, one can use a very general theorem of conservation ([10], th. 25, § 34), according to which the initial Lagrangian density can be replaced by the quantity $\sum[-T_5^5]=\sum r+A_\mu\sum J^\mu$ (*) the quantity r which is associated to each phenomenon can therefore be interpreted as the *classical Lagrangian density in the absence of electromagnetic field*; we have thus proved the *principle of minimal interactions*, according to which the electromagnetic interactions can be obtained by adding to the gauge-invariant Lagrangian $\sum r$ the term $A_\mu\sum J_\mu$.

Let us recall that the original pentadimensional Lagrangian density is by hypothesis gauge-invariant.

3. – Five dimensional wave equations.

We have just seen that the principle of relativity in five dimensions gives a *complete explanation of classical electrodynamics*; can quantum electrodynamics be treated in the same manner?

This problem raises great technical difficulties; in particular, because the theory unifies gravitational field and electromagnetic field, and that one cannot be quantified without the other. It seems that the very formulation of quantum mechanics should be renovated if it is to find its expression within the frame of differential geometry.

We shall limit ourselves to a partial problem, that of a single particle in a given exterior field (electromagnetic and gravitational); to this effect, we shall write the wave equations which satisfy the principle of relativity in five dimensions on the manifold U, and we shall interpret them in Kaluza's approximation.

0-spin particles (see ([14])). – We write on the five-dimensional manifold the linear invariant equation

$$\pentagon\,\varphi + a\varphi = 0\,, \tag{13}$$

where φ denotes a real wave function, $\pentagon$ Dalembertian in five dimensions, and a denotes a real constant.

In a *standard map* (see the Appendix), φ has the period 2π in the fifth dimension; we can therefore write the Fourier expansion

$$\varphi = \sum_Z \varphi_Z \exp[iZx^5]\,, \tag{14}$$

where Z denotes an integer; the φ_Z are complex functions of x^μ; φ_Z and φ_{-Z} are complex conjugates; φ_0 is real.

(*) The same theorem, when applied to ordinary relativity in the static case, shows that the Lagrangian density can be replaced by T_4^4, that is by the *density of energy*; it thus justifies the principle of stationary energy to characterize equilibrium.

([14]) J.-M. Souriau: *Coll. Intern. C.N.R.S., Royaumont*, **91**, 293 (1959).

The operator $\bigcirc$ can be calculated by the classical formula

$$\bigcirc \varphi = \frac{1}{\sqrt{|g|}}\, \partial_j [\sqrt{|g|}\, g^{jk}\, \partial_k \varphi] \,. \tag{15}$$

If we make Kaluza's approximation and introduce the *transverse variables* and the A_μ, we have

$$|g| = \xi^2 |\tilde{g}|; \quad g^{\mu\nu} = \tilde{g}^{\mu\nu}; \quad g^{\mu 5} = -\frac{1}{\xi}\sqrt{\frac{\chi}{2\pi}}\, A^\mu; \quad g^{55} = \frac{1}{\xi^2}\left[-1 + \frac{\chi}{2\pi} A_\mu A^\mu\right]; \tag{16}$$

if we introduce these expressions in (13) and (15), we see that each Fourier component φ_Z of φ satisfies the quadri-dimensional equation

$$\square \varphi_Z - \frac{iZ}{\xi}\sqrt{\frac{\chi}{2\pi}}\, [\operatorname{div}[A\varphi_Z] + A^\mu \partial_\mu \varphi_Z] - \frac{Z^2}{\xi^2}\left[-1 + \frac{\chi}{2\pi} A_\mu A^\mu\right] \varphi_Z + a\varphi_Z = 0; \tag{17}$$

if we restrict ourselves to special relativity, and if we introduce the quantities

$$q = \frac{Z}{\xi}\sqrt{\frac{\chi}{2\pi}}\, \hbar; \qquad m = \hbar \sqrt{\frac{Z^2}{\xi^2} + a}\,, \tag{18}$$

this equation becomes

$$\tilde{g}^{\mu\nu}\left[\partial_\mu - \frac{iqA_\mu}{\hbar}\right]\left[\partial_\nu - \frac{iqA_\nu}{\hbar}\right] \varphi_Z + \frac{m^2}{\hbar^2}\varphi_Z = 0\,, \tag{19}$$

which is Klein-Gordon's equation for a particle of charge q, mass m, spin 0, in the presence of an electromagnetic field.

Thus, five-dimensional relativity gives a geometrical origin to the action of a field on such a particle (*); it shows furthermore that electric charge is necessarily an integral multiple of an elementary charge $(\hbar/\xi)\sqrt{\chi/2\pi}$, *which does not depend on the mass of the particle.*

Giving the charge the usual value e (which is for instance the charge of a π^+ meson for which eq. (19) is supposed to hold), we get the numerical

(*) For the orthodox physicist, this theory gives, oddly enough, an argument in favour of general relativity: in quantum mechanics, most of the time the gravitational interactions can be neglected, but not electromagnetic interactions; now the geometrical origin which we attribute to the latter cannot be dissociated from gravitation, in the very form which Einstein gave to it.

In other words, even if we neglect the gravitational constant χ, the electromagnetic field cannot be described penta-dimensionally without introducing a curvature to the universe.

value of ξ:

$$\xi = \frac{h}{e}\sqrt{\frac{\chi}{2\pi}} = \frac{2\hbar}{ec}\sqrt{G} = 3.782\cdot 10^{-32}\ \mathrm{cm} \tag{20}$$

(G is Newton's gravitational constant),

this length is *extremely small* compared to the usual dimensions of particles (which are of the order 10^{-13} cm); this gives an *a posteriori* justification to the initial hypothesis that ξ is so small that the fifth dimension will remain hidden.

The theory thus gives a geometrical origin to the quantification of electric charge, which has no explanation in four-dimensional relativity.

The usual circumstance in quantum mechanics where a neutral particle is described by a real wave function, whereas a charged particle is described by a complex wave function, thus appears quiet naturally; it is easy to verify that the gauge transformations and the charge conjugations (see above) act in the usual way (cf. [10], note 42.18); in particular, the charge conjugation $[x^5 \to -x^5]$ obviously replaces each Fourier component φ_Z by φ_{-Z}, which has the value $\bar{\varphi}_Z$.

It is clear that this theory imposes on a spin-0 particle an infinity of charge states (numbered by the integral values of Z), with a mass spectrum given by the second of formulas (18); but the numerical values show that only one of these charge states can be observed (and of course also the opposite charge state); since the mass differences are of the order $\hbar/\xi \neq 5\cdot 10^{20}$ MeV (*).

However the theory gives no explanation of the existence of charge multiplets which are found among particles which have strong interactions (for instance the π-meson); but no such explanation was to be expected (see the Introduction).

Spin ½ particles (see [14]). – The same method can be applied to spinor fields; it is well known that the spinors of the five-dimensional normal hyperbolic space are the same as those of the Minkowski space (cf. [10], (44.33) and (44.34)); let us consider a spinor ψ which satisfies the invariant equation

$$\gamma^j \hat{\partial}_j \psi + a\psi = 0 \tag{21}$$

with

$$\begin{cases} \gamma^j\cdot\gamma^k + \gamma^k\cdot\gamma^j = 2g^{jk} & \text{(signature } +----\text{)}, \\ \hat{\partial}_j = \text{covariant derivation of the spinors (**) along the direction } x^j; \\ a = \text{const}. \end{cases} \tag{22}$$

(*) Let us note, however, that certain primary cosmic rays have an energy approaching this value.

(**) For the definition of this operation, see for instance [10] (Section **45**) and [15].

([15]) J. A. WHEELER: *Geometrodynamics* (New York, 1962).

The eq. (21) can be developed by replacing by their values the Christoffel symbols and the matrices of Fock-Ivanenko, and one obtains

$$(23)\qquad \gamma^j \partial_j \psi + \left\{\frac{1}{2\sqrt{|g|}}\, \partial_j [\sqrt{|g|}\gamma^j] \tfrac{1}{8} - \gamma^j[\partial_j \gamma_k - \partial_k \gamma_j]\gamma^k\right\} \psi + a\psi = 0\ .$$

We now make Kaluza's approximation and introduce the transverse variables (see Appendix); the intermediate formulae are

$$(24)\qquad \left|\begin{array}{l} \gamma^\mu = \tilde\gamma^\mu;\quad \gamma^5 = \tilde\gamma^5 - \dfrac{1}{\xi}\sqrt{\dfrac{\chi}{2\pi}}\,\tilde\gamma^\mu A_\mu = \tilde\gamma^5 - \dfrac{e}{\hbar}\tilde\gamma^\mu A_\mu\,, \\ \gamma_\mu = \tilde\gamma_\mu + \dfrac{e}{\hbar}\tilde\gamma_5 A_\mu;\qquad \gamma_5 = \tilde\gamma_5\,; \end{array}\right.$$

from which we obtain

$$(25)\qquad \left\{\begin{array}{ll} \tilde\gamma_\mu\cdot\tilde\gamma_\nu + \tilde\gamma_\nu\cdot\tilde\gamma_\mu = 2\,\tilde g_{\mu\nu}; & \tilde\gamma^\mu = \tilde g^{\mu\nu}\tilde\gamma_\nu; \\ \tilde\gamma_\mu\cdot\tilde\gamma_5 + \tilde\gamma_5\cdot\tilde\gamma_\mu = 0 & \tilde\gamma_5^2 = -\xi^2 \end{array}\right.$$

inserting in (23), expanding ψ in Fourier serie

$$(26)\qquad \psi = \sum_Z \psi_Z \exp[iZx^5]$$

and restricting ourselves to special relativity, we obtain

$$(27)\qquad \mathrm{Y}^\mu\left[\partial_\mu - \frac{iZe}{\hbar}A_\mu\right]\psi_Z + \left[a - \frac{iZe}{\hbar}\sqrt{\frac{2\pi}{\chi}}\,\mathrm{Y}_5\right]\psi_Z + \frac{1}{8}\sqrt{\frac{\chi}{2\pi}}\,F_{\mu\nu}\mathrm{Y}^\mu\mathrm{Y}^\nu\mathrm{Y}^5\psi_Z = 0\,,$$

where $\mathrm{Y}^\mu = \tilde\gamma^\mu$ are the usual matrices, and where we have noted

$$(28)\qquad \mathrm{Y}_5 = \frac{\tilde\gamma_5}{\xi} = \mathrm{Y}_1\mathrm{Y}_2\mathrm{Y}_3\mathrm{Y}_4\,.$$

The last term in eq. (27) is completely negligible on the scale of quantum mechanics (in the system of units where $\hbar = c = 1$, the constant $\frac{1}{8}\sqrt{\chi/2\pi}$ is of the order 10^{-33} cm), and we suppress this term. If we now introduce the quantities m and λ defined by

$$(29)\qquad \left|\begin{array}{l} m\lambda = Ze\sqrt{\dfrac{2\pi}{\chi}} - a\hbar\,, \\ \dfrac{m}{\lambda} = Ze\sqrt{\dfrac{2\pi}{\chi}} + a\hbar\,, \end{array}\right.$$

and make the change of variables

$$\psi = \frac{1-i\gamma_5}{2}\varphi_z + i\lambda\,\frac{1+i\gamma_5}{2}\varphi_z\,, \tag{30}$$

eq. (27) becomes

$$\gamma^\mu\left[\partial_\mu - \frac{iZe}{\hbar}A_\mu\right]\psi + \frac{im}{\hbar}\psi = 0\,. \tag{31}$$

We recognize Dirac's equation for a particle of charge Ze, mass m, spin $\frac{1}{2}$, in the presence of the electromagnetic field.

Five-dimensional relativity thus gives a geometrical origin to the electromagnetic interactions of fermions as well as to those of bosons; further we obtain the result that the elementary charge is *independent of spin.*

The *neutrino* case is obtained by taking $Z=0$, $a=0$. The change of variable (30) is then no longer necessary and we obtain (neglecting as before the last term of (27)) the usual equation

$$\gamma^\mu\partial_\mu\psi = 0\,. \tag{32}$$

We now wish to study the interaction of an electron and a neutrino, which we denote respectively by the spinors ψ_{el} and ψ_{neut}.

Among the covariant quantities which can be constructed from these spinors, is the *five-component vector* V:

$$V^j = \overline{\psi_{\text{neut}}}\,\gamma^j\,\psi_{\text{el}}\,, \tag{33}$$

taking into account (24) and (30), the transverse space-time components of this vector can be written as

$$V^\mu = \overline{\psi_{\text{neut}}}\,\gamma^\mu\left[\frac{1-i\gamma_5}{2} - \frac{1}{i\lambda}\,\frac{1+i\gamma_5}{2}\right]\psi_{\text{el}}\,. \tag{34}$$

In the study of β-decays, if we suppose that the electron and the neutrino which are emitted result from the decay of a *charged leptonic current* J, we are led to consider the invariant Lagrangian $V^\mu J_\mu$.

Formulas (29) show that the dimensionless constant λ is of the order $10^{\pm 21}$, the sign in the exponent depending on that of a (note that $2m\sqrt{G}/a = \xi mc/\hbar \neq 10^{-21}$); one of the two terms in the expression (34) of V^μ will then be negligible; the Lagrangian can thus be written as

$$gJ_\mu \overline{\psi_{\text{neut}}} \gamma^\mu \frac{1 \pm i\gamma_5}{2} \psi_{\text{el}} = gJ_\mu \overline{\frac{1 \mp i\gamma_5}{2} \psi_{\text{neut}}} \gamma^\mu \psi_{\text{el}} \,, \tag{35}$$

where g is a coupling constant.

Now it turns out that experiment leads one to precisely such a form of Lagrangian: the *maximum violation of parity*, as expressed by SALAM, LANDAU, LEE, YANG, etc., is thus a consequence of five-dimensional relativity.

From the Lagrangian (35), we see that the emitted neutrino is an eigenvectors of γ_5 (such a neutrino is said to be in a pure state of *chirality*, or else that it is a *two-component* neutrino); the evolution eq. (32) shows that it will remain in such a state.

But two-component spinors have no existence in five-dimensional geometry: it is thus necessary to introduce a *second two-component neutrino*, of opposite chirality. Now, we know that a second neutrino has indeed been found experimentally (*).

We thus see that the principle of relativity in five dimensions is perhaps destined to play a role in the description of elementary particles and their interactions, even for those which it does not « explain ».

A final remark should be made: if we look algebraically for the set of linear operators which commute with the γ_j, we see that it is isomorphic to the *quaternion* field Q; this allows us to give to the space of Dirac's spinors a structure of Q-vectorial space, such that the γ_j will be Q-linear (**).

The multiplication of ψ by a fixed arbitrary quaternion obviously does not change eq. (21); in other words, the vectorial space of its solutions is also quaternionic. This property no longer holds in the quadri-dimensional eq. (31) of the electron, because of the particular role plaid by the quaternion i (see (**)); but it remains in eq. (32) of the neutrino; we recognize the gauge transformation of Pauli-Gürsey.

* * *

The author wishes to thank Professor L. MICHEL for fruitful discussions on the interpretation of the theory, and Professor J. MANDELBROJT for help in the English version of the article.

(*) This is of course independent of the existence of the corresponding antineutrinos; in the theory it may be that the existence of the antineutrinos can be deduced from the fact that it is necessary to introduce simultaneously two Dirac's spinors in order to have the spin representation of the *complete* Lorentz group in five dimensions.

(**) See [10], Sections **43** and **44**. In fact, the complex number i that we have in the formulas (14), (17), (19) can be considered as a particular quaternion.

Appendix

The interpretation of fields in the theory of Jordan-Thiry.

To describe the five-dimensional manifold U, we use co-ordinate systems or *maps* which we consider as applications of R^5 on U.

Thus if

$$X = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \\ x^5 \end{bmatrix},$$

is an element of R^5, the point M of U which has these co-ordinates in the map F will be denoted by

$$M = F(X)\,.$$

We shall only use *standard maps* of U, that is those such that x^5 have the period 2π.

M being a point of U, the fibre $\widehat{M}$ which passes through M is the closed curve obtained when x^5 alone varies (for example from 0 to 2π); if we note

$$\hat{X} = \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix},$$

we see that the fibre $\widehat{F(X)}$ depends only on the map F and on $\hat{X}$; we shall write

$$\widehat{M} = \hat{F}(X)\,.$$

We then see that the set $\hat{U}$ of the $\widehat{M}$ can be considered as a four-dimensional manifold, of which $\hat{F}$ is a map (since $\hat{X} \in R^4$); it is this manifold $\hat{U}$ which is interpreted as the usual space-time.

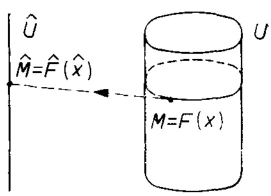

Fig. 1.

Let us note that two different standard maps F and F^* can be equivalent in $\hat{U}$:

$$\hat{F} = \hat{F}^*\,.$$

In this case, if we write $M = F(X) = F^*(X^*)$, it is a consequence of the axioms of the theory that

$$\begin{cases} \widehat{X^*} = \hat{X}\,, & (i.e.\ x^\mu = x^{*\mu} \text{ for } \mu = 1, 2, 3, 4)\,, \\ x^{*5} = \pm x^5 + f(\hat{X})\,, & \end{cases}$$

5135

The co-ordinate transformation which makes one pass from F to F^* is inter preted as a *gauge transformation*, if we have the $+$ sign before x^5, and as *charge conjugation* if we have the $-$ sign. It does not modify the co-ordinates of space-time x^μ.

An arbitrary geometrical field defined at the point M can be represented by the components Y_q in a map F; the *variance* of the field will be caracterized by the transformation formulas of Y_q when one passes from one map to another; fields therefore have both a *space-time variance* and a *gauge variance.*

It can be useful to associate to each standard map F a map $\widetilde{F}$, equivalent to F in U, and which is *transverse*, that is such that the hypersurfaces $x^5 = \text{const}$ are *orthogonal* to the fibres; except in some rare cases, this condition can be realized only in one point; it can be proved that the components Y_q in F can be expressed in terms of the transverse components $\widetilde{Y}_q$ in $\widetilde{F}$ and of the quantities $\mathscr{A}_\mu = g_{\mu 5}/g_{55}$; in a gauge transformation $[x^5 \to x^5 + u]$, the transverse components $\widetilde{Y}_q$ are *invariant*, the $\mathscr{A}_\mu$ transform according to the formula

$$[\mathscr{A}_\mu \to \mathscr{A}_\mu + \partial_\mu u] \text{ (*)} .$$

The field equations can therefore be expressed by means of gauge-invariant variables (the transverse components $\widetilde{Y}_q$) and of the potentials $\mathscr{A}_\mu$; the equations will then be automatically covariant both in space-time transformations, gauge transformations and charge conjugations.

(*) This method can easily be generalized to the case where the variance involves second derivatives, or even derivatives of a higher order (for instance in the case of connections); see [10], Section **41**.

RIASSUNTO (*)

Si esamina l'ipotesi che l'universo U sia un complesso di Riemann a cinque dimensioni, che soddisfa alcune condizioni topologiche globali. Si postula l'esistenza di un principio di relatività che tratta in modo uguale le cinque dimensioni di U; le leggi che soddisfano a questo principio hanno una descrizione approssimativa in un complesso spazio-tempo a 4 dimensioni $\hat{U}$; ciò dà la possibilità di confrontarle con la descrizione usuale delle leggi sperimentali. Così, se si estende alla quinta dimensione l'invarianza della relatività generale, si ottiene l'*elettrodinamica classica*: le equazioni di Maxwell, la conservazione dell'elettricità, le forze elettromagnetiche, ecc. Similmente, l'estensione in cinque dimensioni dell'invarianza delle equazioni d'onda ci porta automaticamente a termini elettromagnetici, quali si osservano effettivamente; si trova, per esempio che la carica elettrica è un multiplo intero di una *carica elementare* che non dipende nè dalla massa, nè dallo spin. Tra le altre conseguenze della teoria troviamo l'*invarianza di gauge*, e la *coniugazione delle cariche*; la *massima violazione di parità* nei decadimenti η; l'esistenza di *due neutrini* di chiralità opposta.

(*) *Traduzione a cura della Redazione.*