GRAVITATIONAL ACCELERATION OF SPINNING PARTICLES

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The principles of Classical Mechanics can be formulated by using a differential 2-form σ, the exterior derivative of which, $\nabla\sigma$, being zero.

The equation of motion is written $\sigma(dy)(\delta y) = 0 \quad \forall\, \delta y$ [or, in a shorter way, $dy \in \ker(\sigma)$], where y denotes a point in the manifold of initial conditions.

In the case of a particle of spin s and mass m, in special relativity, the structure of the Poincaré group allows one to caracterise completely y and σ ; y is defined by the position x^μ of the particle, its unitary speed U^μ and its spin sJ^μ ; U^μ and J^μ satisfy the following conditions

$$U_\mu U^\mu = 1 \quad ; \quad J_\mu J^\mu = -1 \quad ; \quad U_\mu J^\mu = 0 \; ;$$

the form σ is given (without any external field) by

$$\sigma(dy)(\delta y) = m\left[dx_\mu \delta U^\mu + \delta x_\mu dU^\mu\right] - s\, d\Omega^\nu_\mu \Omega^\rho_\nu \delta\Omega^\mu_\rho$$

where $\Omega_{\mu\nu}$ denotes the adjoint of the tensor $U_\mu J_\nu - U_\nu J_\mu$.

The Bargmann-Michel-Telegdi equation is readily obtained by adding to σ a term corresponding to the electromagnetic interaction.

In order to take in account the gravitational field, it is natural to adopt general relativity and to replace the derivatives d and δ by covariant ones (denoted by $\hat{d}$ and $\hat{\delta}$). But the exterior derivative of σ is non longer 0 ; Bianchi identities can be used to show that the condition $\nabla\sigma = 0$ is satisfied by adding a new term :

$$\sigma(dy)(\delta y) = m\left[dx_\mu \hat{\delta} U^\mu + \delta x_\mu \hat{d} U^\mu\right] - s\, \hat{d}\Omega^\nu_\mu \Omega^\rho_\nu \hat{\delta}\Omega^\mu_\rho - \frac{s}{2}\Omega^\sigma_\rho R^\rho_{\mu\nu,\sigma} dx^\mu \delta x^\nu$$

where $R^\rho_{\mu\nu,\sigma}$ denotes the curvature tensor.

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Applying the principle of geodesics to a spinning particle consists in neglecting the length $\frac{s}{m}$ ($2\ 10^{-11}$ cm in the case of the electron); expanding to the first order in $\frac{s}{m}$ the condition $dy \in \ker(\sigma)$, one obtains the equations of motion of the spinning particle in a gravitational field

$$\frac{dx^\mu}{d\tau} = U^\mu \quad ; \quad \frac{\hat{dU_\mu}}{d\tau} = -\frac{s}{2m} R^\rho_{\nu\mu,\sigma} \Omega^\sigma_\rho U^\nu \quad ; \quad \frac{\hat{dJ_\mu}}{d\tau} = \frac{s}{2m} U_\mu R^\rho_{\nu\xi,\sigma} \Omega^\sigma_\rho U^\nu J^\xi$$

(τ = proper time); therefore, one obtains a term describing a proper acceleration of gravitational origin ; the particle trajectory is no longer a geodesic.

In the Schwarzschild static case, the curvature tensor is small everywhere : the maximal propre acceleration at the surface of a spherical body is of order of $\frac{s}{m} \rho G$ (ρ = body density ; G = gravitational constant), that is to say 10^{-17} cm sec^{-2} for an electron in the neighborhoud of the Earth.

On the other hand, the gravitational waves, which propagate a non negligible curvature, can accelerate in a sensible way the spinning particles.

If the particles are electrically charged, they must radiate a continuous spectrum of electromagnetic waves ; this effect could be considered as an explanation of pulsar radiation (the source of gravitational waves being, for instance, the fast rotation of a system of neutron stars).

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