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Physics and Geometry

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Differential geometry, the contemporary heir of the infinitesimal calculus of the 17th century, appears today as the most appropriate language for the description of physical reality. This holds at every level: The concept of "connexion," for instance, is used in the construction of models of the universe as well as in the description of the interior of the proton. Nothing is apparently more contrary to the wisdom of physicists; all the same, "it works." The pages that follow show the conceptual role played by this geometry in some examples—without entering into technical details. In order to achieve this, we shall often have to abandon the complete mathematical rigor and even full definitions; however, we shall be able to give a precise description of the connection of ideas thanks to some elements of group theory.

1. WHAT IS DIFFERENTIAL GEOMETRY?

We know that Euclidean geometry is based on the notion of equal figures—two figures are said to be equal if they can be brought into coincidence by an operation called *displacement.*

By noticing that the Euclidean displacements form a *group*, in the sense today quite classically defined by Evariste Galois, F. Klein gave a generalization to the concept of geometry; more precisely, a *classification of geometries* (Erlanger program, 1872). A geometry is given by a set E (the space) and a group G of transformations of E, which plays the roll of displacements.

Let us take as example an ordinary surface S, for instance a sphere. We can choose as group G for the set of all transformations of S that are continuous and have continuous inverse (such transformations are called homeomorphisms); the geometry associated with this choice of G is by definition the *topology* of the surface.

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The *differential geometry* of S is obtained if we choose a subgroup G of the group of homeomorphisms; we require every element of G to be differentiable, which means that the functions involved have partial derivatives of all orders. The elements of this restricted group are called *diffeomorphisms* of the surface S. These diffeomorphisms have for instance the property of transforming two curves traced on S that are tangent to each other into two curves that are also tangent, and two osculating curves into osculating curves; this means that these so-called "contact properties" belong to differential geometry.

The differential geometry is not concerned only with surfaces; the general spaces involved are called *manifolds*. Curves are manifolds in one-dimension; surfaces are manifolds in two-dimensions; and, in general, one considers manifolds of arbitrary n-dimension.

Is it so hard and so abstract to imagine a manifold in four-dimensions for instance?

Let us consider the set D of all straight lines of ordinary space. We can convince ourselves easily that a line depends on 4 parameters and that it is natural to choose 4 numbers in order to identify it (for instance the coordinates x_1, y_1 and x_2, y_2 of its intersection with 2 parallel planes P_1 and P_2). It is clear that this system of coordinates is not valid for all straight lines. For instance, it will become necessary to choose another set of planes and to define new coordinates in order to identify a straight line in the case when it becomes parallel to the initial planes; and it can well be imagined that for certain lines one has to take a 3rd pair of planes.

The situation is entirely analogous in the case of a sphere, for instance the surface of the Earth. We can choose coordinates on the Earth; this is necessary when we want to construct a map. In order to represent all of the Earth, we have to construct an atlas composed of several maps (for instance a Mercator projection and 2 maps of the polar regions).

This same language is used in order to *define a manifold V of arbitrary n-dimension*: One requires the existence of an atlas of V, constituted of "maps" or "coordinate systems," which represents all of V. The only condition that is required is that the formulas for changing coordinates, which effect the transition from one map to another, should be differentiable.

So, the set D of all straight lines of ordinary space *is a manifold*; the dimension of D is 4, because we need 4 coordinates in order to identify a straight line d, *considered as a point of D*.

2. A LITTLE BIT OF HOMOTOPY

We have just studied the manifold of straight lines, which we have called D; we can associate with it another manifold L, *having the same four-*

dimension, the points of which will be *oriented lines* (can you find an atlas of L composed of 6 charts?). We shall use L further as the manifold of the *light rays*.

Since a line d has two orientations, l' and l'', say, we can associate the two points l' and l'' of L with the same point d of D (see (Fig. 1). We say that L is a *covering space* of D.

The manifold L is made out of a single piece, in the sense that two points of L can always be joined by a continuous curve (let us think of the manipulations of a stick!); we say that L is *connected.*

We have just found a connected covering space of the manifold D. For other connected manifolds, there are no such connected coverings; we say that such manifolds are *simply connected.* For instance, a sphere S is simply connected and the manifold L of light rays, too.

Let l be an oriented line, that is a point of L. Let us denote by $R(l)$ the same line but oriented in the other direction. A little thought would convince us that R is a *diffeomorphism* of L; it is clear that the composition $R \circ R$ of R with itself the identity I on L. Consequently, R and I form a group H of diffeomorphisms of L, and we can say that the manifold D is the quotient of the manifold L by this group H; we shall write this as $D = L : H$.

This situation is general: If V is a connected manifold, there exists a simply connected manifold W, a group H of diffeomorphisms of W, such that $V = W : H$. Together with precise conditions, which of course can not be described here in detail, this construction is standard: The manifold V defines entirely the manifold W (which is called the *universal covering* of V) and the structure of the group H (called the *homotopy group* of V).

Of course, these concepts can scare one by their abstraction, *but in physics one comes across various phenomena that can not be understood*

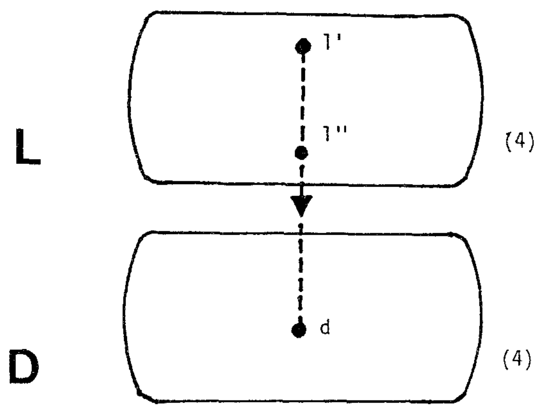

Fig. 1. Covering Space.

without the help of the key given by homotopy theory. We shall see several examples of this.

The displacements of Euclidean space constitute, as we have just said, a group; but they form also a manifold of six dimensions (it is not too difficult to choose 6 parameters in order to parametrize a displacement, for instance the coordinate of the image of a point and 3 rotation angles. Such objects that are simultaneously groups and manifolds (with a few rules of compatibility between the two structures) are called *Lie groups* (after Sophus Lie, a Norwegian mathematician). Lie groups appear today as one of the most important achievements of pure mathematics and of theoretical physics.

What does the theory of homotopy teach us when applied to a Lie group G? if G is not simply connected, its universal covering G' is still a Lie group; the homotopy group H of the manifold G is a commutive subgroup of G'.

Let us consider the example of the group G of Euclidean displacements. In this case, the homotopy group H has 2 elements—the identity and another one; the universal covering G' is consequently different from G. A question arises then: since classical Euclidean geometry is associated with G, does there exist a supergeometry associated with the group G'? This is an inverse problem (knowing the group, to find the space), which has been resolved by Elie Cartan.[3,4]

Contrary to what one might believe, it is a concrete problem. The material objects that most resemble mathematical points are obviously the particles of microphysics. Experiments show that they come in 2 kinds, ruled either by the classical geometry of the group G or by the supergeometry of the group G'; they are called, respectively, bosons and fermions.

Go around an ordinary object, a boson for instance. When you have completed the circuit, you will again find yourself at they place of departure and nothing has changed. But if you do the same thing for a fermion, the situation will be different! In order to recover the initial state, you will be obliged to circle the fermion twice. Why twice? Simply because the homotopy group H is a two-element group.

Nothing is more common than a fermion (for instance an electron); and nothing contradicts more our habits of thought than this double circuit paradox. However, this is not a crazy idea; the corresponding experiment has been performed and has given the result just announced about half a century after its theoretical prediction.

This kind of fact shows that our space intuition is not firm and that it is necessary to have recourse to concepts of differential geometry, for instance homotopy, in order to understand correctly the real world.

3. WHAT IS A GEOMETRICAL OBJECT?

In order to have a useful answer to this question, it is indispensable to commit to memory two algebraic definitions, even if we don't like that.

(a) Let G and H be two groups, M a mapping from G into H (if you prefer, a function defined on G, with values in H); we say that M is a *morphism* if it has the following property:

$$M(gg') = M(g)\,M(g')$$

valid for any elements g and g' in the group G.

(b) Let F be a set and consider the group $F!$ of all permutations (or bijections) of F. We call *action of a group G on a set F* every morphism of the group G into the group $F!$ of the permutations of F.

Let us return now to Euclidean geometry; G is here the group of displacements. Let us consider a geometric figure, for instance a triangle (A, B, C). If g is a displacement, the new triangle $(g(A), g(B), g(C))$ will be called the *image* of the triangle (A, B, C) through the displacement g, and we can write this:

$$I(g)(A, B, C)$$

Three lines of calculation are enough to verify that the operation "image" (we just called it I) is an *action* of the group G on the set of all triangles [in the sense (b) above].

As a generalization, *every time we will have defined the action of a group G on a set F*, the elements of F will be called "geometric objects." Some of those objects, such a triangles, can be figures composed of space points; but there are other objects, and they are just as important for the physical sciences.

So, for example, mechanics, electromagnetism, and crystallography have led us to create and use the following concepts: *free vectors* (*axial* or *polar*), sliding vectors, torsors, and tensors; all these are geometrical objects for the group G of displacements. The *spinors*—which have been constructed of course in order to describe particles with spin—are geometrical objects for the group G' of the supergeometry studied in Section 2.

The old infinitesimal calculus is mainly based on the notion of points tat are infinitesimally close to a given point. There are again *geometric objects*, not only for Euclidean geometry, but also within the broader framework of differential geometry. The terminology that is used is that of a *tangent vector* v to a manifold X in a point x. If g is a diffeomorphism, the action of g transforms v into a new vector, tangent to X at the point $g(x)$.

One can define the product of v by a number, the sum of two tangent vectors, and these operations are invariant under diffeomorphisms. As a consequence, all the possibilities of linear algebra will allow us to create new geometrical objects attached to a point X: *linear forms* (or *covectors*), *tensors* of any order p, antisymmetric tensors or p-forms, *densities*, *tensor densities*, *capacities*, *orientations*, and so on. All these objects have an important role to play in mathematical physics.

Let us finally quote some other geometric objects which have somewhat subtle definitions, but are just as useful: *germs*, *jets*, *connexions*, and so on.

4. SYMMETRIES

Let us consider a figure that is symmetric in the usual sense of the word, for instance a triangle (A, B, C) with 2 equal sides: $AB = AC$.

The symmetry of this figure is the symmetry with respect to the height from A; it is a displacement g of the plane, characterized by the properties $g(A) = A$, $g(B) = C$, and $g(C) = B$. In other words, the image under g of the triangle (A, B, C) considered as a set of 3 points (without ordering) is the triangle itself; if we denote by f the figure constituted by this triangle, we have consequently $g(f) = f$.

Let now f be any figure or, more generally, a geometrical object. We can still call *symmetries* of f for all the elements g of the group G that satisfy this equation $g(f) = f$; this subgroup is called the *symmetry group* or *stabilizer* of the object f. We will see several examples of this.

5. FIELDS AND RELATIVITY

Consider a manifold X, and a function associating with each point x of X a tangent vector at X. Such a function is called a *vector field* of X; in an analogous way one defines tensor fields, p-forms, etc., for all the categories of objects that we quoted in Section 3.

We know that the field theory is a fundamental branch of classical mathematical physics (examples are fluid mechanics, the theory of elasticity, the Maxwellian electromagnetic field, etc.).

A fundamental remark: If g is a diffeomorphism of the manifold X, then the image under g of a field is again a field of some nature; consequently *fields themselves are objects of differential geometry.* As any other geometrical object, a field hs consequently a *symmetry group* (see Section 4).

Let us consider the example of a *crystal*, considered as a continuous unbounded medium—that is, as a *field* in the sense just mentioned. We know *a priori* that its symmetry group is a subgroup of the group G of Euclidean displacements; a courageous mathematical analysis shows that there are exactly 230 types of subgroups of G that are suitable (*crystallographic groups*). We have consequently a fundamental classification of crystals; it allows us to recognize *a priori* those that can give rise to a given type of physical phenomena (for instance piezoelectricity, bi-refrigence, etc.). This classification remains relevant even when one has at one's disposal a more refined model that takes into account the atomic architecture of the crystal.

We call a manifold *Riemannian* if on it we are given a tensor field $g_{\mu\nu}$ satisfying the conditions $g_{\nu\mu} = g_{\mu\nu}$ and $\det(g_{\mu\nu}) \neq 0$.

Albert Einstein has shown that the most exact representation of gravitation is given by such a Riemannian structure on 4-dimensioned spacetime, and that the $g_{\mu\nu}$ can be considered as gravitational potentials. This theory—*general relativity*—has been confronted with observation uninterruptedly for more than 60 years. Thanks in particular to the achievements of NASA, it is today a physical theory that has given rise to predictions satisfied with the highest precision (of the order 10^{-12}).

We should notice that there exists and *inverse relationship* between gravitational effects, on the one side, and the symmetry of the fields, on the other; in particular, the total absence of gravitation (Minkowski space) gives rise to maximum symmetry, defined by a group in 10-dimensions), called the *Poincaré group*. The Poincaré group defines of course a geometry on spacetime; if one allows only this geometry as relevant physical reality, then one is doing *special relativity*.

We see clearly the relationship between both general and special relativity: The two theories are associated with two geometries and consequently with two groups; the second one is a subgroup of the first. Consequently, a geometrical object in general relativity induces automatically an object of special relativity, but the inverse problem does not have a standard solution; there are physical quantities having the same geometrical status in special relativity but different ones in general relativity. This difference is in itself a basic item of information about the nature of the phenomenon, as important as the information given by the dimensional analysis.

Let us take the example provided by the *thermodynamics of continuous media*. There are a large number of variables intervening here: specific mass, energy, momentum, velocity, stress, strain, speed of deformation, temperature, heat, entropy, etc. How do we put some order into this jumble? The principle of general relativity gives an answer—unusual but efficient—to this problem. According to this principle, physical phenomena are represented by geometrical objects associated with the group of diffeomorphisms: This allows a fine classification in which, for instance, temperature and velocity are associated to define a tangent vector in spacetime (according to prescription already proposed by Max Planck); heat and entropy production define a vectorial density etc. Recognizing these facts, we obtain immediately information on possible relationships between these quantities (it is for instance out of the question to add a vector and a vectorial density! The result would not be the same any more after a diffeomorphism). One gets to see all that one could hope to obtain by studying thermodynamics from the point of view of differential geometry.

General relativity gives us universal rules for classification and selection of quantities and physical laws, at least on the macroscopic level. A theory that has been suitably formulated in this framework can be easily translated into special relativity and then into classical mechanics.

For some years now, the continuous media specialists have invoked a certain "principle of material indifference," which is considered necessary in order to write down correctly the laws of behavior of materials. But one has never been able to formulate this principle exactly. However its formulation is within reach: It suffices to accept a detour through general relativity. The same method allows us also to treat simultaneously macroscopic electrodynamic phenomena, such as magnetization, magnetostriction, gyromagnetic effects, etc.

6. SYMPLECTIC MANIFOLDS

Symplectic geometry must resemble Euclidean geometry to start with: One considers a manifold X, endowed with a tensor field $\boldsymbol{\sigma}$ (it will replace the metric tensor g), which is assumed reversible $[\det(\sigma_{\mu\nu}) \neq 0]$ and *flat* (i.e., there exists an atlas of X where the $\sigma_{\mu\nu}$ are constants). But then we replace the symmetry condition $g_{\mu\nu} = g_{\nu\mu}$ by the antisymmetry condition $\sigma_{\mu\nu} = -\delta_{\nu\mu}$. One can prove that these conditions are compatible only if the dimension of the manifold X is an even number.

The symplectic structure was discovered in 1811 by Joseph-Louis Lagrange[4]; the covariant and contravariant components of the tensor $\boldsymbol{\sigma}$ are the "brackets" and the "parentheses" of Lagrange. Their discovery is the result of deep investigation into the structure of the equations of mechanics.

The manifold to which Lagrange gives a symplectic structure is the set of solutions of equations of motion belonging to a dynamical system—we shall simply say the *space of motions*.

This theory, developed in *Mecanique Analytique*, a classic par excellence, was however not really understood by the contemporaries and followers of Lagrange; Poisson and Hamilton, for instance, transmitted it only in a truncated form. In particular, the symplectic structure was defined on "phase space," a disastrous choice which leads at the same time to the disappearence of the global properties and relativistic properties of mechanics.

A century later, Elie Cartan, completing the works of Henri Poincaré on integral invariants, reinvented the symplectic form (the "absolute integral invariant"), and the true dimensions of Lagrange's work reappeared progressively.

In its present geometric form, this theory has not been developed before the 1950s. It is easy to write down the symplectic form σ with the help of initial conditions: positions, velocities, and time[8,20]; the forces thus acquire the status of components of σ, which fixes their variance uncer arbitrary change of coordinates. In the case of rotating reference frames, for instance, the centrifugal and Coriolis forces appear spontaneously. It is clear that the flatness condition of σ (see above) imposes restrictions on the forces; in the case of the electromagnetic forces, one thus sees the appearance of Maxwell's homogenous equations. Let us indicate finally that the "principle of virtual work" can be obtained by a truncation of σ (forgetting the time variations).

The symplectic structure appears also in all spaces whose points are solutions of a problem in variational calculus—the calculus which was developed, as one knows, by Euler and Lagrange, and which concerns the determination of maxima and minima.

For instance, a straight line is the shortest distance between two points. It follows from this that *the set L of oriented straight lines are a symplectic manifold.* This structure is particularly important in optics: The instruments (mirrors, telescopes, photographic lenses, etc.) are characterized by the transformation (input) → (output) that they impose on light rays. This is a point transformation of the manifold L of rays. One of the properties of this transformation is that it commutes with the object which we denoted by R in Section 2: This is the exact formulation of the *principle of exact return of light.* Another property of this transformation is that it preserves the symplectic structure; this theorem of Lagrange follows simply from the fact that the laws of optics can be obtained from a variational principle, the principle of Fermat.

Cleverly exploited by the physicist Ernst Abbe, this fundamental result lies at the basis of the modern methods of calculation of optical instruments (relations of aplanetism, eikonal, etc.). Abbe was one of the co-founders of the Zeiss enterprise in Iena.

The possibility of using a variational principle in classical mechanics (Pierre de Maupertuis and William Hamilton) suggested quite soon an analogy of structure between classical mechanics and optics; this analogy was exploited in the early quantum theories, in particular in the wave mechanics of Louis de Broglie.

However, the variational formulation of mechanics is *less powerful* than its symplectic formulation: It is not manifestly covariant under the Galilei transformations (see Section 7), it requires an arbitrary choice of potentials and vector potentials, and finally there are systems for which *it is impossible*—while the symplectic formulation still exists. For instance, this is the case for spinning particles (see Section 7); the classical equations of motion for these objects[2] were written down and used only after their description in quantum mechanics became known.

7. DYNAMICAL GROUPS

Let X be a symplectic manifold (Section 6). The symmetries of the field σ (which has been defined in Sections 4 and 5 above) are called *symplectomorphisms* of X; the group of symplectomorphisms defines the *symplectic geometry* of X.

Let us study a free dynamical system in space (for instance any molecule or the solar system). Each possible motion of the system is obviously an object of Euclidean geometry; consequently, the group of Euclidean displacements will act on the symplectic manifold X of the motions.

In all the known cases, one notices that this action is obtained *by symplectomorphisms*; consequently, one of the essential physical characteristics of the system is a *morphism*

$$G \to (\text{symplectomorphisms of } X)$$

where G is the group of Euclidean displacements. Every time a Lie group G is endowed with such a morphism, we say that G is a *dynamical group* of X (technically one requires in addition to some differentiability conditions).

Every "symmetry" of the system, defined by the existence of a dynamical group, is associated with a *conservation law*. This is shown by a fundamental theorem established by the mathematician Emmy Noether in 1918 (the theorem, as written down originally, concerns groups of one-dimension and variational systems; we shall study the general case).

If G is a dynamical group of a symplectic manifold X, we can associate with every point of X a new quantity called a *moment*; this moment itself is a geometrical object belonging to the space of torsors of the group G (some indications for connoisseurs: a torsor is a 1-form of the Lie algebra of G; the action of G is coadjoint). The number of components of the torsor is equal to the dimension of G.

We shall forgo here a mathematical definition of the moment, giving only some examples.

Let us take the case of a dynamical system that is free in space and the group of displacements; then the moment is an entity of 6 components consisting of the *linear momentum* and the *angular momentum* of the system. By their construction, these quantities remain fixed when the system evolves. For this reason, they are called "constants of motion," "conserved quantities," or "first integrals."

Let us now consider the manifold L or oriented lines or light rays, the symplectic structure of which we have studied in Section 5. There, too, the group of displacements are a dynamical group; the associated moment exists, but what good is it?

In fact, the mathematicians have known for a long time the 6 components of this moment: They are the "Pluckerian coordinates" of a line, used in elementary geometry. Anatural question arises: Is it possible to give them a mechanical interpretation in terms of linear and angular momenta?

This interpretation is, in essence, the one given by Einstein in 1905,[6] when he invented a particle that is a carrier of light (the photon). Einstein assigned explicitly to this particle various mechanical properties, such as linear momentum, and he showed that the photoelectric effect could be correctly interpreted as a transfer of quantities associated with a photon to the target matter.

Consequently, it is the symplectic quantities, and particularly the moment, that appear in a transfer between apparently disjointed domains of physics, such as optics and mechanics. This suggests of course that the symplectic structure plays a *universal role.* In fact, *all conserved quantities that are usually considered, in mechanics or in the rest of physics, can be obtained as moments of a suitable chosen dynamical group G.* We shall verify this with new examples.

A conservative system is obviously invariant under the group of "temporal translations"—whose operations retard or advance all motions by the same time. The moment corresponding to this group is the *energy.*

A free dynamical system is a geometric object, geometric not only under the Aristotle group (Euclidean displacements + temporal translations), but also under a group of 10-dimensions (the Galilean group), which can be obtained by completing the group of Aristotle by the transformations of Galilei—transformations which consists in the addition of one and the same initial velocity to all the points of the system. An application of Noether's symplectic theorem then predicts that the motion of the center of mass is rectilinear and uniform; this result is sometimes known as the "principle of inertia" (it was formulated for the first time by Gassendi).

In order to establish this result, Newton had to introduce a special principle into mechanics, the equality of action and reaction. Here *this is not necessary*; the Galilean symplectic in variance is a sufficient principle.

Every time that we are in the presence of a dynamical group G, there arises the problem of "equivariance": We know the action of G on the symplectic manifold X, on the one hand, and on its torsors, on the other. The question is to determine whether the moment mapping, inserted between those 2 actions, leaves them compatible. One of the subtleties of the situation is that the moment mapping is not completely defined; one can add an arbitrary constant to it.

Luckily the mathematicians have constructed a tool called *cohomology*, which allows us to analyze this situation. This theory enables us to know whether equivariance is or is not possible, and it allows us to diminish the arbitrariness of the additive constants. Even better, it allows us to define new quantities—the *cohomology classes*—which will be physically interpreted.

When applied to the Galilei group—that is, to classical mechanics—cohomology gives essential result.

Out of the 10 quantities constituting the moment, only one, to wit the *energy*, contains an arbitrary irreducible additive constant.

Correlatively, there appears a new quantity, the *class of cohomology* of the system; this is simply the *mass*.

Everybody who has taught physics knows that the concept of mass is not very intuitive; this difficulty is related to the subtle status that we have just attributed to this quantity. The question is not only one of a mathematical artifice: We shall see that cohomology is an effective tool for the analysis of some fundamental facts.

Given any system with nonzero mass, we can show that there exists a dynamical group larger than the Galilei group and having the 14-dimensions; this allows us to decompose the energy into a sum of 2 conserved quantities: kinetic energy (of the center of mass) and internal energy. Similarly the angular momentum is a sum: orbital angular momentum + internal angular momentum. There are consequently 14 conserved quantities.

It is true that most of these results can be obtained without the use of the symplectic structure, but here they acquire a universal character which they did not have before; they are valid for all Galilean dynamic systems—even those that cannot be interpreted as a system of interacting mass points (and for which the Newtonian principle of equality of action and reaction cannot be formulated, e.g., for a system that contains magnets).

One can make the transition from classical mechanics to relativistic mechanics without making any change to the symplectic structure—it suffices to replace the Galilei group by the Poincaré group (which was defined above in Section 5). It is the differences of structure between the two groups that generate the qualitative differences between the two mechanics.

For instance, the "symplectic cohomology" of the Poincaré group is zero; it follows that there is no arbitrary constant in the relativistic energy—and that there are no fixed relativistic mass.

For a given observer, the same dynamical system can be studied either in classical mechanics or relativistic mechanics; there exists simple geometric procedure which allows us to identify the *moments* obtained in the two theories. Among the 10 identities thus obtained one finds also the celebrated formula of Einstein,

$$E = mc^2$$

which relates the *relativistic energy* E and the *Galilean mass* m. It is significant that Einstein justified this formula only by rather obscure arguments; the geometrical technique appropriate for this demonstration was not available in 1905. It may be mentioned that this obscurity persists is most modern treatises on relativity.

Let us return to classical mechanics, and consider a point that is attracted or repelled, according to an arbitrary law, by a fixed point O. This system clearly has spherical symmetry around O; the associated momentum is the angular momentum about that point. However, in the particular case of a *Coulomb field*, there is another conserved quantity (the Laplace-Lenz vector); this corresponds to a *hidden symmetry*, and the dynamical group associated with it is composed of the rotations in a 4-dimensional Euclidean space which remains rather mysterious.

This interpretation was guessed in 1926 by Wolfgang Pauli,[16] in a work on the spectrum of hydrogen; this paper has historically been at the origin of quantum mechanics. It is this particular symmetric group that gives rise to the degeneracy between energy—levels of hydrogen—and consequently to the energy *shells* for electrons in an arbitrary atom (in the "hydrogenoid" approximation). These energy shells explain the old notion of chemical *valence*, such as the quadrivalence of carbon, which is a fundamental property for all of organic chemistry and consequently for life itself.

In *nuclear physics*, the "shell" models are in competition with the collective "droplet" models. The latter can be constructed also from a dynamical group (consisting of linear volume preserving transformations of space) and from the properties of the associated moment (S. Sternberg and G. Rosensteel).

To end, let us given some example of the inverse procedure: Knowing a Lie group G, find a symplectic manifold X such that G is a dynamical group of X. Under certain conditions, this problem has a regular solution, the *Kirillov construction*—which has to be completed in order to take cohomology into account.

Thus the Poincaré group is sufficient to define dynamical relativistic systems in an "abstract" way; these mathematical models are ready to accomodate the *elementary particles* that really exist.

The models are indexed by two numbers, the *rest mass* and the *spin*; if the rest mass is zero, a third quantity is required, *viz.*, the *helicity*; it appears as a spatial orientation that can take two values (we can think of the turning senses of a corkscrew).

All these quantities can be *observed* and *measured on real particles*. For instance, mass and spin are part of the "identification card" of the particles. The photon, for example, has mass zero and spin 1. Indeed, there exist two kinds of photons which are polarized circularly to the right or to the left, depending on the value of their helicity; we can select them in natural light with the help of a quartz prism (rotational bi-refringence of Fresnel).

Progressively experiment has uncovered other quantities that are carried by real particles (isotopic spin, hypercharge, strangeness, color, flavor, etc.); simultaneously it has shown that those particles can be grouped into *multiplets*: doublet of nucleons, triplet of pions, octet of baryons, triplet of quarks, etc.

We can associate with each of these multiplets a *symplectic manifold*, which can be constructed by Kirillov's method, starting with a Lie group called a *gauge group*. The new quantities observed in this way are precisely the *moments* associated with this group.

These gauge groups allow us to classify particles and to define their dynamics; they offer a reasonable hope that some day we shall better understand the structure of matter.

8. THE PREQUANTUM LEVEL

We owe to Elie Cartan a general theory of p-forms (see Section 3); the two essential procedures of this theory are the *differentiation* of forms (that generalize the classical operations of gradients, divergence, and curl), and the *reduction* (which in some cases allows one to characterize a form with the help of a manifold of lower dimension).

Let us consider a manifold Y endowed with a 1-form (in other words a tensor field ω_μ). It can happen that ω is irreducible, but that its derivative $d\omega$ is reducible; this situation is known as a *contact structure*. In this case, one can show that the dimension of manifold Y is odd (i.e., given by $2n + 1$), and that the form $d\omega$ can be reduced, in general, giving a *symplectic* structure to a manifold X of dimension $2n$ on which Y is projected (Fig. 2).

The set of points y of Y that are projected into the same point x of X is

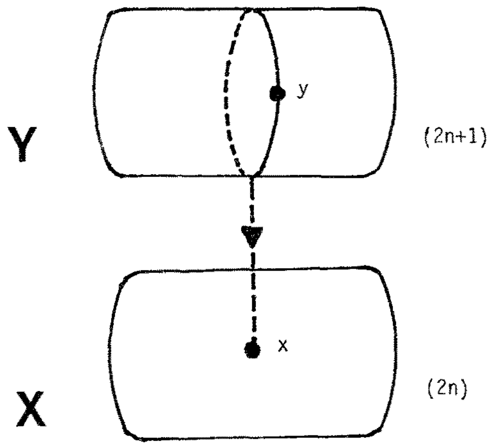

Fig. 2. Prequantization.

a curve. Let us assume that this curve is *closed*, and consider the circulation h on anyone of them:

$$h = \oint \omega_\mu \, dy^\mu$$

We can show that h is a constant (independent of the choice of the curve). The case which will interest us here is where h is equal to *Planck's constant* $h = 6.6262 \times 10^{-27}$ gm cm^2 sec^{-1}. We shall say that Y is a *quantum manifold.*

Of what use is this new structure? Is is to be employed to describe some natural phenomena. We shall in fact state a new principle of physics: *every dynamical system in reality is associated with a quantum manifold Y.* In Sections 6 and 7 we have described various dynamical systems with the help of a symplectic manifold$X$; *this manifold X has to coincide with a manifold obtained from Y by the construction of Fig. 2.*

Consequently the "classical" description at the level of X is *incomplete*, and we have to reconstruct a quantum manifold Y starting with X. This purely geometric problem is called *prequantization* of X.[13] Theoretically this problem is completely solved: We know how to formulate conditions that allow us to determine whether there is no solution, one solution or several solutions.

Before we examine examples, let us make a remark: This problem is consistent in dimensional analysis, because Planck's constant h and the symplectic form σ defined by Lagrange (Section 6) both have the dimension of an *action*, that is to say ML^2T^{-1}.

The *elementary particles* are associated with symplectic manifolds (see Section 7); we can show that the problem of prequantization can be solved only if the *spin* s of the particle is an integral multiple of the number $h/4\pi$; and *this happens for all known particles*: $s = h/4\pi$ for protons, neutrons, and electrons; $s = h/2\pi$ for photons; etc.; we observe a multiple going up to a factor of 6. In all these cases consequently the quantum manifold Y exists and one can show that it is unique.

For other systems, the problem of prequantization of X can have *several solutions*: the geometry even tells us exactly how many solutions there are: as many as there are *morphisms* (a, Section 3) of the *homotopy group* of X (Section 2) into the *torus* (that is, group of rotations of the circle). Theoretically this number can be always determined.

Accordingly, for a charged particle circulating around a straight solenoid, there is *an infinity* of possible prequantizations (one can identify each of them by a point on the torus); the one "chosen by nature" is characterized by the intensity of current in the solenoid.[9] At first sight, this situation is very classical; in fact it is *paradoxal* to common sense, because the magnetic field created by the infinite solenoid is *zero* and consequently should not have any effect on the particle. However, the effect just described has been observed in interference experiments (Bohm–Aharonov effect). This is a typical case where one's imagination is insufficient and where it is necessary to use differential geometry.

Let us now consider a system of n *identical particles*; here the homotopy group consist of permutations of particles; one can show that there are exactly 2 possible prequantizations.

Experience shows that nature chooses one or other prequantization, depending the type of particles, *bosons* or *fermions* (see Section 2); this choice is physically manifested by collectives properties:

Fermions (for instance, electrons) satisfy the *Pauli exclusion principle*, which particularly allows the existence of the *solid state*; thanks to this principle we are unable to go through walls or floors. Bosons (for example photons), on the other hand, can all congregate into a single collective state. This is what happens in the coherent light of lasers; it is also this property of bosons that helps explain the phenomena of superconductivity and superfluidity.

We have just sketched a "bestiary" of quantum manifolds that are observed in nature. In order to proceed to the "zoological" level, one has to find some specific properties of quantum manifolds Y that are effectively observed. One of these properties is the following: Y should be able to accomodate a geometric object, called *polarization*,[13] or even a more precise object called *polarizator*.[21] Christian Duval[5] has deduced from this hypothesis rules that are observed in all the examples known until now:

spin of a particle of zero mass has to be equal to the product of $h/4\pi$ by 1, 2, or 4; some gauge multiplets (see Section 7) are excluded, and this forbirds for instance the quarks to group themselves in pairs; etc.

9. DOES QUANTUM MECHANICS EXISTS?

Let us consider a quantum manifold Y, the structure of which is characterized by a tensor field ω_μ (see Section 8).

The *symmetries* of ω_μ (in the precise sense of Section 4) are called *quantomorphisms*. These quantomorphisms constitute necessarily a group Q which define the *geometry* of Y. It is to this geometry that, for instance, the objects considered in Section 8, polarizators, belong.

Q is not a Lie group because *its dimension is infinite*; however one can consider the *universal covering* Q' (see Section 2), which we will call the *quantum group*. Q' defines of course a "supergeometry" of Y, which will be used in quantum physics.

The correspondences between groups we dealt with until now can be summarized by the following diagram:

$$Q' \to Q \to S$$

where the two arrows are morphisms (see Section 3) and where S is the group of symplectomorphisms of X (Fig. 2). We shall now reinterpret elementary physical facts in this new description of nature.

The quantities which can be physically measured—called *dynamical variables* or *observables*—are functions defined on the manifold X; however, they can also be defined as *subgroups* of Q' (subgroups of one-dimension). In Section 7 we have followed the inverse path: going from the subgroup and associating in with a dynamical variable, the *moment*. Universality of this notion is required here: *The only observable quantities are now moments*, each of which is linked to a natural symmetry.

In classical mechanics (Sections 6 and 7) we have characterized the effective state of a system by a *point* of the manifold X (a "motion" of the system). We shall now modify this point of view, by making a more direct connection with the groups: a *quantum state* will be a function defined *on the quantum group* Q' and satisfying certain conditions that we cannot write down here in detail (the states constitute a convex set of "positive-definite" functions).

In classical mechanics an observable f in a motion x of a system, took the value $f(x)$, and this was the value one imagined as being measured experimentally. Here, however, we have a new situation: An observable and a quantum state define mathematically a *spectrum* (the Fourier transform of the function "quantum state" on the group "observable"). Physically this spectrum is the only available information concerning the results of measurements. Such is the *probabilistic interpretation of quantum mechanics*. The fact that the quantum group is *noncommutative* makes it possible to obtain systematic lower bounds on the *width* of this spectra; in such a way one generalizes Heisenberg's *uncertainty relations*.

Let a be an element of the group Q' and let m be a quantum state; the function $a(m)$, defined on Q' by the formula

$$a(m)(b) = m(a^{-1}ba)$$

is again a state; this formula defines an *action* of Q' on the set of states; hence the quantum states are themselves geometrical objects, which belong apparently to the supergeometry of Y. In fact, commutation relations show that the state $a(m)$ depends on a only through its projection on the group S of symplectomorphisms. From this it follows that the *quantum states are objects of symplectic geometry*—in other words, *they belong to classical mechanics*. This is a precise formulation of the "correspondence principle" between quantum mechanics and classical mechanics.

Quantum mechanics is not yet a closed mathematical theory; there still exists incoherences, and one can only apply it with certainty in domains which have been solidly tested, as in quantum chemistry, for instance; on the other hand, the theoretical basis of nuclear physics still remains to be constructed.

The method sketched here (the "*geometric quantization*") does not escape completely these difficulties; the quantum states, which are in principle functions defined on the whole group Q', are in practice mostly only defined on more or less large subgroups. Hence one can only predict the spectra of *some* observables, but in these cases the theoretical predictions conform well to the experiments.

Utilizing several tools of harmonic analysis (Gelfand–Naimark–Segal construction and Stone theorem) one can exhibit a Hilbert space H. One is able to describe in this way a quantum state by a *vector* of H (which is only defined *modulo* a phase) and to associate with a classical observable a *self-adjoint* operator. In this way a connection is established between this "geometric quantization" and the usual procedures of quantum mechanics.

We should remark that the definition of states as functions on a group is more general than their definition by a vector since it contains the case of the mixed states of quantum statistical mechanics (Gibbs states, Hartree-Fock approximation of quantum chemistry, etc.).

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