DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **19**, Number **3**, November **2007**

Website: http://aimSciences.org

pp. **595–607**

ON GEOMETRIC MECHANICS

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Abstract. This survey paper introduces the reader to the origins of the Geometric Mechanics theory and traces its subsequent history.

1. **Applied Mechanics.** I have studied problems of vibrations and stability which arise in aeronautics as well as other technologies; this work has allowed me to elucidate criteria of stability which may be formulated in easily calculated algorithms from theoretical data, or from experiments. These have been used systematically in various domains (subsonic and supersonic airplanes, navigational instruments, etc.). This work comprised my *thèse de Doctorat d'État*, *"Sur la stabilité des avions"* [14]. *See references* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

2. **Theoretical Mechanics.** The discoveries of Lagrange (*Mécanique Analytique*, 1788-1813) can be interpreted today in the language of global differential geometry: the *collection of motions* [1] of a dynamical system is a manifold with an antisymmetric flat tensor (a symplectic form) whose contravariant and covariant components are respectively the "(round) brackets" and "(square) brackets" of Lagrange. This structure contains all the pertinent information of the state of the system (positions, velocities, forces, etc.).

The *symmetries* of a symplectic manifold induce a mapping which I have called the *moment map*, which takes its values in a space attached to the group of symmetries (in the dual space of its Lie algebra); to the moment map is associated a specific mathematical object, the *symplectic cohomology*.

In the case of a dynamical system, the moment map is a *constant of the motion*; this result generalizes such earlier notions as the Hamiltonian, the invariant theorem of Emmy Noether. It plays an important role in diverse branches of mechanics: general theorems, reduction of partially symmetric systems (Marsden, Weinstein, Arnold), classification of completely integrable systems, etc. There also exist applications to celestial mechanics, such as the so-called Robbin-Smale-Souriau theorem on relative equilibria.

2000 *Mathematics Subject Classification.* 70-02, 74-02, 80-02, 81-02, 82-02, 83-02.

Key words and phrases. Applied mechanics, Theoretical mechanics, Mechanics of deformable solids, Statistical mechanics, Thermodynamics, Geometric quantization.

This article is an English translation by David H. Sattinger of Jean-Marie Souriau's paper *Autour de la mécanique géométrique*.

[1] It is on this manifold of motions which Lagrange worked in his *Mécanique Analytique* (*see* [88]); a tradition probably going back to Hamilton suggests instead the use of "phase space." But the definition of phase space depends on the choice at each instant of a frame of reference; it is therefore inappropriate even for describing galilean relativity.

This is why we adopt the point of view of Lagrange: the manifold of motions is the mathematical object appropriate for a relativistic or global study of mechanics.

One of the principles of classical mechanics is that the *Galilean group* is a symmetry group of every isolated dynamical system. It suffices to admit that this symmetry *respects the symplectic structure* in order to obtain a number of results – beginning with the equivalence of *action and reaction*, which is no longer an independent axiom. In the same way one obtains a purely geometric haracterization of *mass*: it is the measure of the symplectic cohomology of the action of the Galilean group. Taking account of this remark, one may apply a theorem which guarantees that the manifold of motions is the Cartesian product of a 6 dimensional manifold (the motions of the barycenter) with a reduced manifold. This barycentric decomposition applies also to the conserved quantities: the 10 components of the moment (energy, impulsion, etc.) are associated with 4 others; the kinetic momentum is a sum: orbital moment + internal moment; in the same way, energy is decomposed into kinetic energy + internal energy.

All the details of classical mechanics thus appear as geometric necessities. Moreover, this geometric formulation goes beyond classical mechanics *stricto sensu*; it permits, for example, the construction of a precise model of *particles with spin.* A spinning particle is not a top but a specific Galilean object; the spin is characterized by a proper kinetic momentum independent of the motion. Thus spinning particles possess a purely classical description, i.e., prior and underlying quantum mechanics.

On the other hand, certain "accidental" constants of the motion are associated with hidden symmetries; e.g., in celestial mechanics the Laplace-Lenz vector is connected to a somewhat mysterious symmetry of the *Kepler manifold* [66], the space of constrained motions of a material point in a Coulomb field. Pauli and Fock have presented this as an $O(4)$ symmetry which prolongs the natural $O(3)$ symmetry; I have been able to calculate it exactly, as well as the symmetry of the conformal group $O(4, 2)$, associated with the conserved quantities defined by Bacry and Gyorgyi.

I have also been able to construct a complete system of globally analytic first integrals of the *two body problem* [83], which remain regular just as well away from collisions as from the elliptic-hyperbolic transition; they embed the space of motions in an algebraic manifold, whose complex analytic prolongation is still symplectic. All the details of *relativistic mechanics* are also obtained from the symplectic structure, by simply changing the symmetry group: (the Lorentz-Poincaré group is substituted in place of the Galilean group). The fact that the symplectic cohomology is then null explains the lack of conservation of mass in relativity; there exists a precise procedure passing from the Galilean momentum to the relativistic momentum which contains among other things Einstein's relation $E = mc^2$, etc.

The symplectic structure has the remarkable property to be able, in certain cases, to be reconstructed from its symmetries alone (the Kirillov-Kostant-Souriau orbits); I have been able to construct by this technique some relativistic mechanical models *a priori* for *elementary particles*; geometry alone shows that they must be characterized by an internal mass, an internal kinetic momentum and, in the case of zero mass, a spatial orientation. The particles which one observes are inscribed in this scheme (for example: *photons* in their state of circular polarization, left or right).

The *relativistic dynamics* of these particles results – notably those particles with spin: precession of spin, magnetic resonance, the Bargman-Michel-Telegdi equations (with a correction for which the level is below the precision of the measurements).

Experiments have led to the discovery of other conserved quantities attached to particles: isotopic spin, color, charm, etc.

In 1960 [27], I obtained the first model of the *baryon octet* constructed from a unitary representation of a *symmetry group of strong interactions* (the group $SO(8)$).

The group $SU(3)$ is a sub-group of $SO(8)$ (via the adjoint representation). This is a representation of the reduced group which furnished in 1961 the Gell-Mann-Ne'eman model; the development of this line led successively to the *quark model*, then to the *quantum chromodynamics*. In such *gauge theories*, the dynamical system is physically a multiplet of particles; mathematically it is constructed on a symplectic manifold, support of symmetries (strict or approximate).

The symplectic structure remains pertinent to the modeling of more complex objects; thus the exceptional symmetry of the Kepler manifold (see above) explains the existence of layers in *hydrogenoïds atoms*; the moment map is applied to the study of collective models (nuclear drop model; S. Sternberg); it furnishes a generalization of the Hartree-Fock theory (Rosensteel and Rowe, 1981); etc. *See the references* [15, 16, 27, 29, 31, 32, 33, 40, 43, 45, 46, 47, 48, 49, 59, 63, 66, 69, 83, 88]*; and, for synthetic expositions,* [50, 62, 85] *or* [74, 81, 82].

3. **Continuous media, Statistical mechanics, Thermodynamics.** I developed, beginning in 1970, a description of matter by an "Eulerian tensor-distribution" which is valid as well for *condensed states* as for *continuous media*; it permits a uniform treatment of the dynamics of mechanical structures on our scale (shells, plates, beams, cords) and that of particles – including particles with spin. This description is complementary to that obtained beginning with symplectic symmetries (§2).

This method is applied in classical mechanics as in special or general relativity. It structures interactions of matter with the *electromagnetic field* as well as with the *gravitational field*. In the case of particles it gives a geometric interpretation of the mass, spin, electric charge, and the electric and magnetic moments. In conjunction with measure theory (see §§3 and 4), it describes the statistical effects: thus a distribution of particles with spin gives a realistic model of macroscopic magnetism: *magnetization*, *gyromagnetic effect*, *magnetostriction*.

The method also permits a description of the individual or statistical behavior of particles subjected to a *gauge field* (Duval, Weinstein, Guillemin-Sternberg). The equations thus obtained are in general *non-predictive* (which differentiates them absolutely from the symplectic description which contains within it the equations of evolution; but they associate the conserved quantities with the symmetries of the field: thus non-Noetherian quantities).

Let us consider now an isolated dynamical system, described by a symplectic manifold. This structure must persist in the presence of an external field; consider the gravitational case.

Since the system is isolated, so that it is not subject to an external gravitational field, we are therefore in the conditions of special relativity. Imagine however an infinitesimal perturbation of the gravitational potentials in a bounded region of space-time. If the system crosses this region, it will be subject to a *gravitational scattering* which may be described by an "eikonal" function. In applying the *principle of general relativity* (in other words, the invariance of the gravitational gauge), one states that the eikonal defines an Eulerian tensor-distribution Θ whose support consists of the world lines of the particles constituting the system; each motion x is thus localized in space-time.

Outside the perturbation zone, the *moment map*

$$\Psi : x \mapsto m$$

is factored via the localization

$$x \mapsto \Theta$$

into

$$x \mapsto \Theta \mapsto m,$$

the mapping $\Theta \mapsto m$ being defined by the non-Noetherian conservation laws indicated above.

This abstract result will be a key for the interpretation of statistical mechanics. In classical statistical mechanics a *state* is constituted by a solution of the Liouville equation on the phase space, the partition function. This definition is simplified by Lagrange's viewpoint that a statistical state is simply a probability measure ρ on the manifold of motions.

Let us use the "localization" $x \mapsto \Theta$. By simple linearity the mean value of Θ in the statistical state ρ is again an Eulerian tensor-distribution T; if ρ is sufficiently smooth, T will be a symmetric solution of the Euler-Einstein equations of continuous media; T will thus give a space-time interpretation of the statistical state in terms of density, specific impulse, and stress [2]. The factorization lemma above shows that the non-Noetherian conserved quantities associated with the tensor T are the components of the mean value in the state ρ of the variable moment $m = \Psi(x)$.

It is a question therefore of the mean value of the spectrum of the moment m, i.e., of the image under Ψ of the measure ρ. By construction this spectrum is a constant of the motion. Suppose now that the system undergoes, for a certain time, a *dissipative process*. Experience shows that the spectrum of the moment is modified – the spectrum of the energy, for example, appears "smoothed out"; the statistical state itself has thus been modified by the dissipation. How? We do not know.

It is therefore not legitimate to invoke the mechanical theorem of live forces to justify something that would resemble the conservation of energy in thermodynamics. But *general relativity* will show that dissipative evolution cannot modify these spectra in any manner *whatsoever*. Indeed, the tensor T, which we have interpreted in non-dissipative (time) periods as a characteristic of gravitational *susceptibility*, is at the same time a local source of the gravitational field (the second member of Einstein's equations).

Now this second member exists as well during the dissipative periods – even if we do not know how to calculate it – and automatically satisfies Euler's equations (as consequences of the Bianchi identities applied to the first equation). T – which before and after the dissipation coincides with the statistical mean value of the moment – therefore possesses an Eulerian interpolation during the dissipation.

The globally conserved non-Noetherian quantities associated with T take the same values before and after the dissipation – precisely because they are conserved. Now we know that in the two statistical states they coincide with the mean value of the moment. Whence the result:

[2]From this the interpretation of pressure as a manifestation of the stochastic character of velocity in the kinetic theory of gasses. The stress in a solid medium may account for quantum effects.

The spectra of the ten components of the moment are generally modified by the dissipative processes; but the mean values of these ten spectra are conserved [3]. The first of of these spectra is that of the *Energy*; the conservation of its mean value is precisely the *first principle of thermodynamics.* This principle is thus fallen from its primitive status and reduced to the state of a necessary consequence of the invariance of the symplectic structure in gravitational gauge transformations. Besides it is completed by 9 other thermodynamical conservation laws. This method known as "Souriau Scattering" is easily extended to the case of the electromagnetic field (conservation of the mean electric charge); it is also extended to the general case of gauge fields (Duval, Guillemin-Sternberg).

The *second principle of thermodynamics* is independent: it indicates that the entropy S increases during a dissipation; here we mean entropy in the sense of Clausius-Boltzmann, which is a function of the statistical state ρ. If therefore a state possesses, *for a given mean value of the moment*, greatest entropy, it *will not be subject to dissipation.* These states, if they exist, thus represent the *terminal state* of dissipation. They are indexed by a parameter β with values in the Lie algebra of the Lorentz-Poincaré group; they generalize the *Gibbs equilibrium states*, β playing the role of temperature.

The invariance with respect to the group, and the fact that the entropy S is a convex function of β, imposes very strict, *universal* conditions – i.e., independent of the system considered. For a large class of systems, for example, there exist necessarily a *critical temperature* beyond which no equilibrium can exist. In the cases where an equilibrium exists, it generally consists of a *rigid rotation about the barycenter*, etc.

These purely theoretical results are evidently confirmed by numerous astronomical examples: the Earth and the starts rotating about themselves; dissipative evolution imposes a solid rotation on the central regions of the galaxies, which itself can lead to a gravitational instability of the "quasar" type; the Clapeyron relations extend to the geometrical-dynamical quantities [4], etc.

One can, if one wishes, interpret β as a space-time vector (the *temperature vector of Planck*), giving to the metric tensor g a null Lie derivative. This suggests describing the dissipative processes by a *temperature vector* β which is no longer compelled by this condition; the corresponding Lie derivative of g, the "friction tensor," becomes the source of the dissipation.

One obtains in this way a *phenomenological model of continuous media* which presents some interesting properties: the temperature vector and entropy flux are in duality; the positive entropy production is a consequence of Einstein's equations; the Onsager reciprocity relations are generalized; in the case of a fluid in in the non-relativistic approximation, the model unifies *heat conduction* and *viscosity* (equations of Fourier and Navier). *See references below* [17, 19, 20, 28, 37, 44, 53, 54, 55, 56, 61, 63, 65, 67, 69, 73, 76, 77, 78]*, and for synthetic expositions, see* [38, 3, 81].

4. **Differential geometry and quantum physics.** In his "Principles of Quantum Mechanics," Dirac considered a classical dynamical system; he postulated that one could associate to each *observable* an *operator* on a Hilbert Space $\mathcal{H}$, this correspondence supposedly being linear, and satisfying certain commutation conditions.

[3] These mean values are thus "memorized" by the gravitational field.

[4] They apply, for example, to the pair: inertia momentum – angular velocity.

Taken literally, these principles can have but a heuristic value; they are at the same time contradictory (they must be weakened in order to obtain a coherent theory) and incomplete (implicit complementary hypotheses arise when one applies them in concrete cases).

The search for a theory mathematically coherent and physically achievable, which would constitute a "rational quantum mechanics," is the program of *geometric quantization.*

I have given a construction which resolves the first of the "problems of Dirac" – in 1962 for the case of a Lagrangian system, and in 1965 in the general case – today called "pre-quantization." One considers a space Ξ fibered into circles over the manifold X of motions, equipped with a connection 1-form whose curvature coincides with the symplectic form of X (§2). Ξ is called the *quantum manifold.* One defines $\mathcal{H}$ as a space of functions on Ξ, on which there is a unitary representation of the group G of automorphisms of Ξ. A classical observable γ is identified canonically with a one parameter sub-group Γ of G (the symplectic converse of Noether's theorem). The infinitesimal action of Γ on $\mathcal{H}$ furnishes the operator associated with γ. An equivalent construction was subsequently proposed and published by B. Kostant, with another problematic.

Independent of the problem of Dirac, the existence and uniqueness of such a quantized fibre bundle is a problem in global geometry, whose solution depends on homological and homotopic properties of the space of classical motions. Thus:

- in the case of a particle with spin, the pre-quantization is possible only *when the spin is an integer multiple of* $h/4\pi$ (h = Planck's constant), which is an experimental law;
- the symplectic manifolds associated with particle multiplets are themselves also pre-quantifiable;
- a system of identical particles possesses exactly two pre-quantizations, which may be interpreted physically by the Bose-Einstein and Fermi-Dirac statistics;
- in the case of a particle in motion along a rectilinear conductor, there exist *a priori* an infinite continuum of non-equivalent quantizations; that chosen by nature is determined experimentally by the Aharonov-Bohm effect (P. Horvathy).

All these facts thus suggest that pre-quantization corresponds to a physical reality; but they do not yet furnish the key permitting us to give a rigorous framework to Quantum Mechanics. A first progress has been accomplished using the structure today called *polarization*, which I had introduced in 1953 to interpret the Jacobi theorem [15]. I was able, thanks to polarizations, to construct beginning with symplectic models the wave equations of free elementary particles (Schrödinger, Pauli, Dirac, Maxwell, Yang [50]). Kostant for his part has shown that the polarizations permit the construction of irreducible unitary representations of certain Lie groups.

I have also proposed a more strict mathematical object, the Polarizer, which seems to exist effectively in the case of quantum manifolds associated with material systems (the relativistic wave equations, particle multiplets, etc.; see the works of C. Duval). But the choice of polarization breaks the symplectic symmetry, even in the most elementary case, that of a linear system.

However, in this case, the symmetry must be able to be preserved; this resulted in the works of Stone, Shale, André Weil, V. Bargmann in the 1960's, who have shown the existence of a unitary representation which answers the question. How does one describe in detail this abstract structure with the objectives of symplectic mechanics?

I resolved this problem in 1975 [70], utilizing the Maslov index, a generalization of the Morse index made precise notably by the work of Arnold and Leray. The *Fourier transform* between two n-planes into duality can be interpolated – by regarding them as transverse Lagrangian planes of a linear symplectic space. But the coherent definition of phases in the various spaces comes up against a difficulty: it is a question of *cohomological obstruction*.

This obstruction is resolved by lifting to the covering of the manifold of Lagrangian planes, where the "signature" of Leray becomes the coboundary of the Maslov index. This gives a representation, not of the symplectic group, but of its *metaplectic covering*: the abstract representation of Shale-Weil becomes explicit. This algorithm has been previously used to construct representations of other Lie groups (G. Lion, M. Vergne). The problems of linear quantum mechanics are thus resolved; for example the Schrödinger equation of any harmonic oscillator is integrated explicitly, which produces the *integration formula of Feynman* (with a correction which is necessary beyond the first half-period of motion).

In the preceding examples where quantization is successful, the space of motions was a coadjoint orbit of a certain Lie group. But it concerns only exceptional objects: the non elementary object most discussed, the *hydrogen atom*, is not modeled by such an orbit unless one abstracts the free motions (consequently the ionization and the continuous spectrum are neglected). Again it involves a very simplified model, *spin* and the *magnetic moment* of the proton and electron which constitute the atom are neglected.

In 1985 I constructed a more realistic classical model of the hydrogen atom, which took into account all types of electromagnetic interactions (charge - charge, charge - magnet, magnet - magnet). It involved a symplectic manifold of dimension 16, on which the Galilean group acts naturally (thus a non-relativistic model). The general barycentric decomposition theorem (§2) permits a reduction to a symplectic manifold of dimension 10.

This model has been taken as a test of the heuristic methods of quantization by Duval, Elhadad, and Tuynman (1987). One thus obtains all the observed terms: *spin-orbit coupling* for the electron (fine structure) and for the proton, spin-spin coupling (hyperfine structure), a *diamagnetic* term. But these methods do not give good numerical values for the coefficients – values for which experiments determine with great precision. Worse, the different methods are in contradiction on this point, *even in the non-relativistic approximation*. Quantum physics is still not capable of giving a coherent model of the hydrogen atom which conforms to experiment.

It is therefore tempting to turn toward the methods of quantification emanating from group theory – which requires an enlargement of the frame of Lie groups. I have defined to this effect the structure of "differential group" (1979), subsequently re-analyzed in *diffeological spaces and groups*. The idea is to weaken the axioms of manifolds, replacing the notion of chart with that of "plots," not necessarily invertible, and which does not put in play any particular dimension. One thus obtains a "closed Cartesian" category, in which differentiable mappings of *diffeological* spaces are themselves organized into *diffeological* spaces.

It turns out that the important theorems of differential geometry hold as well in this extended framework – under the condition, of course, that they are correctly reformulated. Homotopy of groups, and more generally of *diffeological* spaces (c.f.

the works of P. Donato and P. Iglesias); *fibrations* and *connections* (P. Iglesias [99]), *differential forms à la* Cartan, and in particular invariant forms of groups on which the coadjoint representation is very naturally extended; etc.

This "*general diffeology*" *appears to furnish* the necessary infinite dimensional objects in diverse physical theories. Thus the set of sections of a fiber bundle (fields) can be equipped with a particular "*diffeology*," the so-called "*controlled diffeology*," which is well adapted to the formulation of the *calculus of variations*; 1-forms of this space of fields (*distibutions*) contains and generalizes *distributions*.

For example the collection L of Lorentzian structures of the space time manifold X can be equipped with the "*controlled diffeology*," as well as the group $Diff(X)$ of diffeomorphisms of X. The action on L of the composition of the connected component G of $Diff(X)$ is a *principal fibration* (in the *diffeological* way!) whose base H shall be called *Physis*. A 1-form on L is a tensor distribution; in order that this 1-form be basic, it is necessary and sufficient that it be Eulerian.

This way is geometricized the duality between the *principles of conservation* of mechanics (which are formulated by the Eulerian nature of the distribution of matter) and the principle of relativity (which postulates the inobservability of the action of the gravitational gauge group G). Matter and geometry in the universe are described by a single point of the cotangent of the *Physis* H.

The Einstein equation itself is formulated by a global 1-form on H (closed and exact), to which must belong the matter-geometry couple. This interpretation generalizes to *electrodynamics*: the distribution of matter is associated with the distribution of current and electric charge – in such a way as to take into account the electrostatic force and the Laplace force: the 10 gravitational potentials $g_{\mu\nu}$ are associated to the 4 electromagnetic potentials A_ρ. The gravitational and electromagnetic gauge groups constitute a semi-direct product; and the coupled Einstein-Maxwell equations define a closed 1-form on the new *Physis* [90].

Diffeology also permits the attainment of other types of objects, whose dimension is no longer infinite, but whose topology is larger. This is the case of the *torus of Denjoy-Poincaré* (quotient of the usual torus by an line of irrational slope), which however is a good diffeological space. Its diffeomorphisms, the fibres of which it is the base, have been classified. This classification puts in play the *arithmetic properties* of the slope (irrational quadratic, diophantine – see the works of Donato and Iglesias). These constructions will perhaps permit us to cast a new light on some problems of theoretical mechanics.

Let us return to the problem of geometric quantification. It happens that all the pre-quantizable symplectic manifolds (hence those which model a concrete physical system) are coadjoint orbits of a diffeological group G. By extension of successful examples of quantization (free elementary particles and the Poincaré group, linear systems and metaplectic representations), one can hope to describe the quantum physics of the system by means of certain unitary representations of such a group.

What are the suitable representations? I proposed in 1986 a definition of "*quantum representations*"; it is founded on the axioms of "*quantized states*" [5]. A quantized state m is a complex function defined on the group G, satisfying a double set of inequalities. These axioms guarantee first a *probabilistic interpretation* of quantum mechanics: the state m associated with all "observables" of G a probability law:

[5]This axiomatic system is applied a priori to a case larger than that of just the symplectic manifolds; there exists, for example, a quantized representation of the Weyl-Heisenberg group in infinite dimension.

in the linear case the *Heisenberg uncertainty relations* are automatically satisfied. The collection of quantized states possessed is a weakly compact convex set; the Krein-Milman theorem therefore permits the generation of this convex set by its extremal points (pure quantized states).

Thanks to the Gelfand-Naimark-Segal construction, every quantized state m can be characterized by the triple consisting of a *Hilbert space* $\mathcal{H}$, a *state vector* Ψ in $\mathcal{H}$, and a *unitary representation* u of G on $\mathcal{H}$. This representation may itself be qualified as "quantum," in the sense that every unit vector Ψ of $\mathcal{H}$ defines a quantized state. The representations associated with pure quantized states are *irreducible*.

By assuming a continuity hypothesis, one associates to each classical observable of the group (defined by a one parameter subgroup) a self-adjoint operator of $\mathcal{H}$ (Stone's theorem); this is with the following properties:

- for each Lie subgroup of G, linearity and the *commutation relations of Dirac* are satisfied;
- if a classical observable is bounded, the *spectrum* of the associated operator admits the same bounds.

The convexity of the space of states produces the "mixed" quantized states whose existence is necessary to *quantum thermodynamics* (Gibbs states) and *quantum chemistry* (molecular orbitals, Gibbs states at absolute zero).

The preceding results thus give a coherent mathematical structure for the usual procedures of quantum mechanics. It is still necessary that there exist *at least* one quantum state; the others are constructed by methods of non-commutative harmonic analysis.

The problem of existence has been resolved – with conclusions conforming to physics – in several interesting cases : the Stern-Gerlach effect, quark model, Heisenberg group in finite or infinite dimensions; research continues in these areas. *On these questions see the book* [50] *and the articles* [15, 16, 33, 40, 43, 46, 47, 48, 64, 70, 71, 72, 74, 80, 81, 82, 84, 86, 87, 89]*; as well as the theses of Duval, Donato, Horvathy, Iglesias, Tuynman.*

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Received for publication June 2007.

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